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IFUSP/P-740

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I.L. Caldas
Instituto de Física, Universidade de São Paulo

H. Tasso
Max-Planck-Institut für Plasmaphysik
EURATOM Association, D-8046 Garching, BRD

Setembro/1988

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I.L. Caldas*
Instituto de Física, Universidade de São Paulo
C.P. 20516, 01498 São Paulo, SP, Brazil

H. Tasso
Max–Planck–Institut für Plasmaphysik
EURATOM Association, D–8046 Garching, BRD

ABSTRACT: The existence of limit cycles of two–dimentional autonomous integrable systems periodically perturbed by delta–peaks is discussed. For that, Poincaré maps are introduced considering analytical solutions between successive pulses.

Nonconconservative non–linear oscillating systems have under certain conditions periodic solutions. Such solutions are called limit cycles and are classified as attractors or repellors according to their stability. They appear in the mathematical analysis of non–linear phenomena(1). Among these are the theory of laser and plasma, biochemical oscillations, circadian rhythms, and many engineering applications in particular electronics. One well known example is the Van der Pol equation which serves as a basic model of self–excited oscillations(2). Another important example is the Brusselator(2).

For large systems of an extended Van der Pol type, Lyapunov functions were found yielding a good estimate of the location of the high dimention limit cycles(3). These estimations can still be done if the systems are perturbed by specific bounded driven excitations.

Various limit cycle system in the plane with an external periodic excitation were numerically investigated from the point of view of chaotic behavior(4). For certain values of the control parameters, some of these systems show mode–locking and period–doubling cascades.

Limit cycles occur also in the descriptions of driven and damped oscillations which can model several physical(2), chemical(5) and biological phenomena(6). Different models have been examined numerically to investigate periodic behavior in two–dimensional autonomous oscillators, periodically forced by discrete jumps in state space(7).

This report investigates the existence of limit cycles (and their bifurcations leading eventually to chaos) of autonomous integrable systems periodically perturbed by delta–peaks. The considered two–dimensional systems were reduced, in the polar coordinate system, to single–variable continous systems periodically driven by a sequence of delta–peaks. The unperturbed systems can be solved between two successive pulses and analytical recurrence relations are obtained for the radial variable and used to define Poincaré maps. By using these maps, it was possible to find analytically modified limit cycles or to show, for high perturbation amplitudes and jump periods, that the solutions either escape or are attracted by a fixed point. However, in general, it is difficult

– even having analytical recurrence relations – to find analytically how the systems behave. A high number of iterations has to be done in order to determine the long time behavior. This can be easily done numerically by using the maps presented here and will be the continuation of this work.

The following two–dimensional nonautonomous systems with periodic forcing were considered:

$$\dot{Y} = MY + \sum_{m=1}^{\infty} c_m \, \delta(t - mT) NY \tag{1}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{aligned} M &= M(\sqrt{y_1^2 + y_2^2}) \\ N &= N(\sqrt{y_1^2 + y_2^2}) \end{aligned} \tag{2}$$

$$c_m = (-1)^m \text{ or } +1 \, ;$$

M and N are non–linear 2 x 2 matrices.

The unperturbed system

$$\dot{Y} = MY \tag{3}$$

is integrable and $Y(Y_0, t)$ can be obtained (between the kicks). The operator M was chosen such that the unperturbed system has limit cycles as attractors and repellors.

If the jumps are repeated with a period T, the system can be described by the map

$$Y_m = FY_{m-1} \tag{4}$$

where F is a non–linear operator depending on the period T and

$$Y_m = \lim_{\epsilon \to 0} \{ Y(t = mT - \epsilon) + \frac{1}{2} c_m N [Y(t = mT - \epsilon)] \, Y(t = mT - \epsilon) + + \frac{1}{2} c_m N [Y(t = mT + \epsilon)] \, Y(t = mT + \epsilon) \} \, . \tag{5}$$

The considered systems can be integrated (for $c_m = 0$) in the polar coordinate system:

$$y_1 = r \cos\theta \qquad y_2 = r \sin\theta \quad . \tag{6}$$

a) For

$$M = \begin{bmatrix} \dfrac{1}{\sqrt{y_1^2 + y_2^2}} - 1 & a \\ -a & \dfrac{1}{\sqrt{y_1^2 + y_2^2}} - 1 \end{bmatrix} , \qquad N = \begin{bmatrix} \dfrac{1}{\sqrt{y_1^2 + y_2^2}} & 0 \\ 0 & \dfrac{1}{\sqrt{y_1^2 + y_2^2}} \end{bmatrix} \tag{7}$$

and

$$c_m = c$$

the system (1) can be reduced to

$$\begin{aligned} \dot{r} &= -r + 1 + c \sum_{m=1}^{\infty} \delta(t - mT) \\ \dot{\theta} &= -a \end{aligned} \tag{8}$$

which leads to the following map

$$r_{n+1} = e^{-T}(r_n - 1) + 1 + c \quad . \tag{9}$$

This gives the radial coordinate r_{n+1} immeadiatly after each kick.

There is still a limit cycle because

$$\lim_{n\to\infty} r_n \longrightarrow 1 + \frac{c}{1 - e^{-T}} \tag{10}$$

b) For

$$M = \begin{bmatrix} f(\sqrt{y_1^2 + y_2^2}) & 1 \\ -1 & f(\sqrt{y_1^2 + y_2^2}) \end{bmatrix} \tag{11}$$

$$N = \begin{bmatrix} \frac{g(\sqrt{y_1^2 + y_2^2})}{\sqrt{y_1^2 + y_2^2}} & 0 \\ 0 & \frac{g(\sqrt{y_1^2 + y_2^2})}{\sqrt{y_1^2 + y_2^2}} \end{bmatrix}$$

and

$$c_m = c \quad ,$$

the system (1) can be reduced to

$$\dot r = f(r) + c\, g(r) \sum_{m=1}^{\infty} \delta(t - mT) \tag{12}$$

$$\dot\theta = -1$$

In the case

$$f(r) = a^2 - r^2 \quad , \tag{13}$$

r = a is a limit cycle of the integrable system (c=0) and

$$r_{n+1} - \frac{c}{2} g(r_{n+1}) = \frac{a \frac{a+r_n}{a-r_n} e^{2aT} - a}{\frac{a+r_n}{a-r_n} e^{2aT} + 1} + \frac{c}{2} g\left[\frac{a \frac{a+r_n}{a-r_n} e^{2aT} - a}{\frac{a+r_n}{a-r_n} e^{2aT} + 1}\right] . \tag{14}$$

In the case

$$f(r) = (a - r)(b - r) \tag{15}$$

r = a is a stable limit cycle and r = b an unstable one of the unperturbed system (c=0).

The map is

$$r_{n+1} - \frac{c}{2} g(r_{n+1}) = \frac{a \left[\frac{b - r_n}{a - r_n}\right] e^{T(b-a)} - b}{\left[\frac{b - r_n}{a - r_n}\right] e^{T(b-a)} - 1} + \frac{c}{2} g\left[\frac{a \left[\frac{b - r_n}{a - r_n}\right] e^{T(b-a)} - b}{\left[\frac{b - r_n}{a - r_n}\right] e^{T(b-a)} - 1}\right] \tag{16}$$

For sufficiently larges T and c there are no limit cycles. However, varying these parameters, it may be possible, by using the maps introduced here, to find numerical regions for which periodic solutions exist and to determine the basins of the eventually found attractors. This will be the continuation of the present work as mentioned before.

ACKNOWLEDGEMENTS

One of the authors (ILC) would like to acknowledge the hospitality of the Max–Planck–Institut für Plasmaphysik and the financial support of FAPESP and DAAD.

* *Partially supported by CNPq.*

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