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INTERACTIONS**

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## DISPERSION RELATION FOR EFFECTIVE INTERACTIONS

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### Abstract:

*It is shown formally that the real and imaginary parts of the effective potential operator for scattering problems obey a dispersion relation even when the hamiltonian is not hermitian. Model calculations are used to demonstrate that the equivalent local potential also satisfies the dispersion relation when absorption is present. It is also shown that the equivalent local potential is insensitive to the bare potential used in the entrance channel.*

### 1. Introduction

The recent discoveries of strongly energy dependent effective interactions in analyses of nuclear scattering data at energies near the coulomb barrier have generated considerable interest. An important unifying concept in this development is the dispersion relation between the real and imaginary parts of the effective interaction.<sup>1,2)</sup> Denoting the formal operator for the effective potential in the entrance channel by  $\hat{V}_{pol}$ , the dispersion relation is written as

$$\text{Re} \hat{V}_{pol}(E) = -\frac{P}{\pi} \int_{E_t}^{\infty} dE' \frac{\text{Im} \hat{V}_{pol}(E')}{E - E'}, \quad (1)$$

where  $E$  is the available energy,  $E_t$  is the threshold energy for non-elastic processes and  $P$  indicates that the principal part of the integral is to be taken.

For the special case where the elastic channel 1 couples via the interaction operator  $\hat{V}$  to a single inelastic channel 2, one obtains<sup>3)</sup>

$$\hat{V}_{pol}(E) = \hat{V}_{12} \frac{1}{E - \hat{H}_2 + i\epsilon} \hat{V}_{21}, \quad (2)$$

where  $\hat{H}_2$  is the hamiltonian in channel 2 and the limit  $\epsilon \rightarrow 0$  is to be taken. The eq. (1) is often recovered by invoking the relation

$$\frac{1}{E - H + i\epsilon} = P \frac{1}{E - H} - i\pi\delta(E - H), \quad (3)$$

in the limit  $\epsilon \rightarrow 0$ .

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The elegance of the dispersion relation in eq. (1) is notable. On the other hand, it says nothing specific about the physical mechanisms which cause the energy dependence. Also the formal operator in eq. (2) is conceptually quite removed from the local optical potential parameterizations which are used to extract the energy dependent effects empirically. For a deeper insight one requires model calculations that explicitly account for couplings to inelastic channels.

Such models for nuclear collisions are invariably based on hamiltonians that are not hermitian. This is not simply because of practical difficulties in dealing with many open reaction channels. Typically, a large part of the reaction cross section is due to the complete fusion process which cannot be modeled by coupling to an open two-body scattering channel, as is assumed in eq. (2). Discussions of the dispersion relation for the effective interaction usually do not consider non-hermitian hamiltonians. For instance, the eq. (3) does not separate the real and imaginary parts of the propagator in this case.

For these reasons, it is interesting to re-examine the effective interaction generated by the type of models which are actually used in practice. In this work we first give a formal derivation of the dispersion relation in eq. (1) which is valid when the hamiltonian is not hermitian. We then present numerical results from model calculations which show how the equivalent local potential follows the behavior of the formal potential operator. This provides an intermediate supporting link between the abstract operator formalism and the energy dependent optical potential parameterizations which are used empirically.

## 2. Dispersion relation

The dispersion relation for the effective interaction can be derived by applying the following mathematical relation,

$$P \int_{-\infty}^{+\infty} dE' \frac{f(E')}{E' - E} = i\pi f(E). \quad (4)$$

This result holds for any function of a complex variable which is analytic on the real axis and in the upper-half  $E$ -plane and vanishes as  $E \rightarrow \infty$ . Expressing eq. (4) in real and

imaginary parts leads immediately to the relations

$$\text{Re}f(E) = \frac{P}{\pi} \int_{-\infty}^{+\infty} dE' \frac{\text{Im}f(E')}{E' - E}, \quad (5)$$

$$\text{Im}f(E) = -\frac{P}{\pi} \int_{-\infty}^{+\infty} dE' \frac{\text{Re}f(E')}{E' - E}, \quad (6)$$

which are referred to mathematically as Hilbert transforms and physically as the Kramers-Krönig relations (see e.g. ref. (4)).

Consider the effective interaction of eq. (2) when the hamiltonian is hermitian. We distinguish this case by writing

$$\hat{H}_2 = \hat{R} = \hat{H}_2^\dagger. \quad (7)$$

Introducing the complete set of eigenstates of  $\hat{R}$ ,

$$\hat{R}|\Phi_r\rangle = E_r|\Phi_r\rangle, \quad (8)$$

we can write

$$\hat{V}_{pol}(E) = \sum_r \frac{\hat{V}_{12}|\Phi_r\rangle\langle\Phi_r|\hat{V}_{21}}{E - E_r + i\epsilon}. \quad (9)$$

Since  $\hat{R}$  is hermitian, its eigenvalues  $E_r$  are real. Consequently, the  $E$ -dependence of eq. (9) satisfies the requirements for eq. (4), the poles being below the real axis. Thus the real and imaginary parts of  $\hat{V}_{pol}$  satisfy eq. (5), which is the same as eq. (1). This argument holds when the coupling interaction  $\hat{V}$  is not hermitian and also when it has an energy dependence that is compatible with the conditions for eq. (4).

For the case of the non-hermitian hamiltonian we write

$$\hat{H}_2 = \hat{R} + i\hat{I}, \quad \hat{H}_2^\dagger = \hat{R} - i\hat{I}, \quad (10)$$

and use the relation

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A+B}, \quad (11)$$

to generate the effective interaction as a series expansion in  $i\hat{I}$ . Taking any term of this series and introducing the eigenstates of  $\hat{R}$ , one sees explicitly that it has poles only in the lower half  $E$ -plane and vanishes as  $E \rightarrow \infty$  (under the assumption that any  $E$ -dependence of  $\hat{I}$  is irrelevant). Thus eq. (4) applies to each term and the dispersion relation linking the real and imaginary parts of the effective interaction also holds when the hamiltonian is not hermitian.

### 3. Equivalent local potential

In this section we investigate numerically the so-called trivalently equivalent local potential. This provides an intermediate link between the formal operators discussed above and the empirical local potentials obtained from analyzing data.

To define the equivalent local potential, it will suffice to consider a two-channel problem. The radial wave functions  $u_i(r)$  satisfy the coupled equations,

$$\begin{aligned} \left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_1(r) - E_1 \right] u_1(r) &= -V_{12}(r)u_2(r), \\ \left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_2(r) - E_2 \right] u_2(r) &= -V_{21}(r)u_1(r). \end{aligned} \quad (12)$$

The boundary conditions at large distances require

$$u_i(r) \rightarrow \delta_{i,1} e^{-ik_i r} + r_{i,1} e^{+ik_i r}, \quad (13)$$

where  $\hbar^2 k_i^2 / 2\mu = E_i$ . Typically one includes absorptive imaginary potentials in  $V_i(r)$  and requires regular boundary conditions on the wave functions at the origin. In the calculations presented below flux loss occurs by virtue of ingoing wave boundary conditions on the wave functions at short distances. From the point of view of the present work, it is irrelevant how one introduces the lack of hermiticity into the model.

Having solved the coupled equations under the appropriate boundary conditions, it is a simple matter to obtain the equivalent local potential for the elastic channel as

$$V_{pol}(r) = V_{12}(r)u_2(r)/u_1(r), \quad (14)$$

so that by construction,

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V_1(r) + V_{pol}(r) - E \right] u_1(r) = 0. \quad (15)$$

Thus if one were given the elastic cross section for this problem and asked to determine the effective interaction which produced it, the correct answer would be the sum  $V_1(r) + V_{pol}(r)$ , where the second term is distinguished by being energy dependent. The result of eq. (14), however, could never be obtained precisely because the elastic cross section at best only determines the reflection coefficients  $r_{1,1}$ .

A large coupled-channels calculation was carried out in ref. (5) to simultaneously describe the fusion and direct reactions in  $^{58}\text{Ni} + ^{64}\text{Ni}$  collisions at energies near the coulomb barrier. The model used ingoing wave boundary conditions to account for the fusion process and contained no additional imaginary potentials. The equivalent local potential for the elastic channel was presented as a function of energy at a radius corresponding to the barrier position. These results are reproduced by the solid curves in fig. 1. The barrier height in this case is about 98 MeV.

It was remarked in ref. (5) that the real and imaginary parts of the equivalent local potential appear to satisfy a dispersion relation. Here we demonstrate that this is in fact

the case. We have parameterized the imaginary part of the effective potential as

$$\text{Im}V_{\text{pot}}(E) = \frac{-A}{\alpha^2 + (E - E_0)^2} + \frac{-B(E - E_0)^2}{\beta^2 + (E - E_0)^2} \Big|_{E > E_0} + \frac{-C}{\gamma^2 + (E - E_1)^2} \quad (16)$$

with  $A = 47.03$ ,  $B = 1.55$ ,  $C = 10.353$ ,  $\alpha^2 = 9.2043$ ,  $\beta^2 = 5.272$ ,  $\gamma^2 = 11.9$ ,  $E_0 = 95.1$  and  $E_1 = 101.7$ . The result is shown by the dashed curve in fig.1. Using this expression in the subtracted form<sup>1)</sup> of eq. (1) generates the the real part as (setting  $E_t = -\infty$ ),

$$\begin{aligned} \text{Re}V_{\text{pot}}(E) &= F(E) - F(\bar{E}), \\ F(E) &= \frac{A}{\alpha} \frac{E - E_0}{(E - E_0)^2 + \alpha^2} + \frac{C}{\gamma} \frac{E - E_1}{(E - E_1)^2 + \gamma^2} + \\ &\frac{B}{(E - E_0)^2 + \beta^2} \left[ \frac{(E - E_0)^2 \ln |E - E_0| + \beta^2 \ln \beta}{\pi} - \frac{\beta(E - E_0)}{2} \right], \end{aligned} \quad (17)$$

with  $\bar{E} = 99.13$  MeV. This formula reproduces the numerical results very well, as shown by the dotted curve in fig.1.

We have made similar comparisons at other distances, including the position where the ingoing wave condition was applied. Here the energy dependence is more complicated than in fig. 1. This causes the parametrization of the imaginary part to be more ambiguous and the reproduction of the real part to be less precise than the example above. It is our conclusion that the dispersion relation is satisfied at other distances to within the uncertainties of our parameterizations. Thus the feature that the dispersion relation of eq. (1) is independent of any spatial properties of the operators is also reflected in the equivalent local potential.

There is another obvious property of the effective potential operator in eq. (2) that is not apparent in the equivalent local potential of eq. (14). The eq. (2) is independent of the hamiltonian in the entrance channel, while eq. (14) depends explicitly on the entrance channel wave function. We have made additional calculations to investigate this point.

The same case was used as in fig. 1 except, for simplicity, only a single inelastic channel was included.

The solid curves in fig.2 show the results for the equivalent local potential at the barrier position for this simplified problem. They are completely analogous to the results in fig. 1 in that ingoing wave boundary conditions have been used in all channels and no absorptive potentials are present. It is again apparent that the dispersion relation is satisfied. The dashed curves in fig. 2 are obtained after including an absorptive potential of the Woods-Saxon type normally used for optical model calculations ( $W = -20$  MeV,  $R_w = 9.35$  fm,  $a_w = 0.63$  fm) in the elastic channel only. Although this obviously distorts the elastic wave and changes the reaction cross section, fig. 2 shows that practically no change occurs in the equivalent local potential.

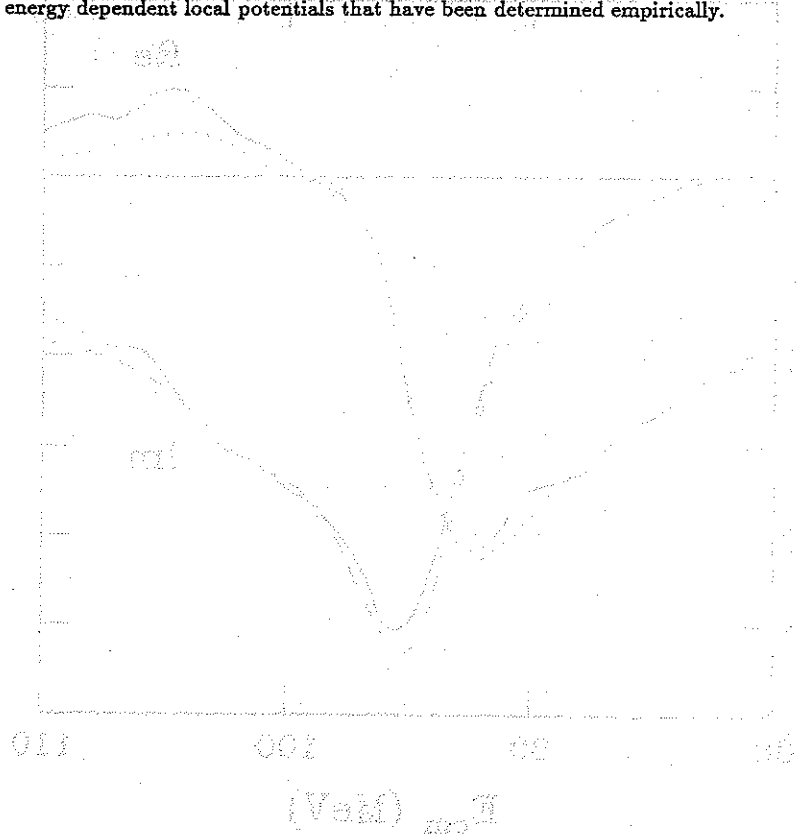
The solid curves in fig. 3 are obtained after introducing the absorptive potential in the inelastic channel only. This changes the equivalent local potential from that shown in fig. 2, but clearly does not alter the dispersion relation between its real and imaginary parts. It is easy to demonstrate this with the fitting procedure described above. Thus, in agreement with the formal operator result, we see specifically that the non-hermitian character of the excited state hamiltonian does not affect the dispersion relation.

Finally, the dashed curves in fig. 3 completes this set of calculations by including the absorptive potential in both channels. They illustrate once more that the equivalent local potential does not depend on the bare potential used in the elastic channel when solving the coupled equations.

#### 4. Conclusion

In this work we have pointed out that the dispersion relation for the effective potential operator holds true when the hamiltonian is not hermitian, as is the case in practically all nuclear reaction calculations. We have also verified by way of examples that the equivalent local potential has the properties expected for the effective potential operator. In particular, it satisfies the dispersion relation when absorption is present. We have also

verified that the equivalent local potential is insensitive to the potential used in the elastic channel when solving the coupled equations. The fact that the local equivalent potential in realistic model calculations obeys the dispersion relation adds theoretical support to the energy dependent local potentials that have been determined empirically.



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### Figure Captions

- 1) The equivalent local potential at the coulomb barrier ( $R_{cb} = 10.8$  fm) for  $^{58}\text{Ni} + ^{64}\text{Ni}$  as a function of the center of mass energy (barrier height  $V_{cb} \simeq 98$  MeV). The solid curves are the results of the coupled-channels calculations in ref. (5). The dashed curve parameterizes the imaginary part with eq. (16). The dotted curve plots the analytic expression for the real part in eq. (17) that is predicted by the dispersion relation.
- 2) The equivalent local potential as in fig. 1 except for a simplified case where only one excited state couples to the entrance channel. The solid curves result when ingoing wave boundary conditions are made in both channels and no absorptive potentials are present. The dashed curves include a typical absorptive potential in the elastic channel only.
- 3) Calculations similar to those in fig. 2. The solid curves result when the absorptive potential is added to the inelastic channel only. The dashed curves include absorptive potentials in both channels.

