IFUSP/P-803

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IFUSP/P-803

PARAFERMIONIC CONFORMAL FIELD THEORY

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To appear in the Proceedings of the IV Jorge André Swieca Summer School.

Setembro/1989

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ABSTRACT

Conformal parafermionic field theories are reviewed with emphasis on the computation of their OPE structure constants. It is presented a simple computation of these for the Z(N) parafermions, unveiling their Lie algebra content.

1. INTRODUCTION

In this lecture it will be reviewed the parafermionic conformal field theories. They appeared in the work of Lepowski and Wilson[1] from the point of view of the representation theory of Kac–Moody algebras, under the name of Z–algebras. Later on, Fateev and Zamolodchikov[2], motivated by some statistical mechanics models, constructed the Z(N) parafermionic field theory. It will be outlined the definition of the Z(N) lattice models together with their expected phase diagrams. Some special points in these diagrams correspond to completely integrable theories[3], in the sense that the Boltzman weights fulfill the Yang–Baxter relation[4]. The Z(N) parafermionic field theory should correspond to these special points[2].

A peculiar fact of the parafermionic field theory is the existence of rational spin chiral currents. Although these models are related to the Wess–Zumino–Witten (WZW)[5] field theories, and so a classical action might be written in terms of fields lying in the group manifold, a direct canonical formalism in terms of those chiral currents is missing. It is not clear if such a canonical formalism exists at all, but as a motivation it will be presented the theory of a single scalar chiral field[6–8] based on the Dirac formalism of constrained systems.

Next it will be defined the Z(N) parafermionic field theory and it will be explained the consistence conditions that led to the computation of the parafermions current algebra structure constants[2].

The general parafermionic field theories[9] will be constructed from Kac–Moody algebras. In fact these theories are the simplest examples of coset G/H models where H corresponds to the Cartan subalgebra associated to G.

The lecture will be closed with an alternative computation of the structure constants of the Z(N) parafermions algebra. We will use the representation theory of level N Kac–Moody algebras[10] and it will become clear that those structure constants are given in terms of group theory factors coming from su(N) level one and su(2) level N.

2. SOME MOTIVATIONS FROM STATISTICAL MECHANICS

The Z(N) lattice models are defined through complex spin $\sigma = \omega^q$, $q = 0,1,\ldots,N-1$ $(\omega = e^{2\pi i/N})$ sitting at the sites of a square lattice L. The Boltzman weights are attached to the links of L and are given by

$$X(\sigma,\sigma') = \sum_{k=0}^{N-1} x_k(\sigma^*\sigma')^k \tag{1}$$

where $x_0 = 1$ and $x_{N-k} = x_k$. Then the partition function is given by

$$Z = \sum_{\langle\sigma(\vec{r})\rangle} \prod_{\vec{r},\alpha} X(\sigma(\vec{r}), \sigma(\vec{r}+\vec{e}_\alpha)) \tag{2}$$

where, as usual, $\langle\ \rangle$ means summing over configurations and $\vec{e}_\alpha$, $\alpha = 1,2$, are the basis vector of L.

These models can be alternatively described by the so called dual Boltzman weights

$$\tilde{x}_k = \left[1 + \sum_{q=1}^{N-1} x_k\, \omega^{kq} \right] \left[1 + \sum_{q=1}^{N-1} x_q \right] \qquad (3)$$

$k = 0, 1, \ldots, N-1$

in terms of which the self–duality condition is stated:

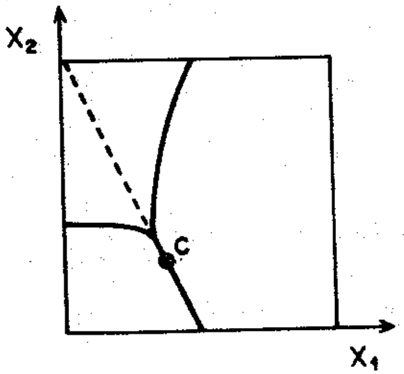
$$\tilde{x}_k = x_k \qquad k = 1,2,\ldots,N-1 \ . \qquad (4)$$

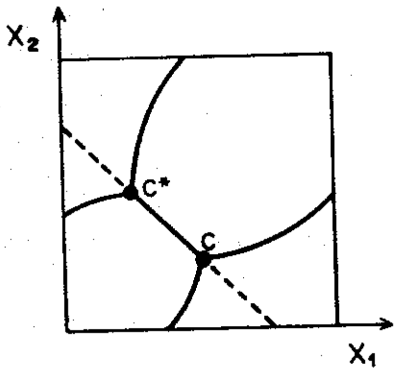
The duality mentioned above generalizes the well known order–disorder duality that is present in the Ising model (N=2) and in the Potts model (N=3). For N=4 and N=5 eq. (4) is given, respectively, by

$$x_2 + 2x_1 = 1$$

$$x_1 + x_2 = \frac{1}{2}(\sqrt{5} + 1) \ .$$

These lines are plotted in the diagrams below by partially broken, partially unbroken lines.





The unbroken lines describe phase transition points and were predicted ten years ago by Alcaraz and Koberle[11]. The points marked C are the ones were one expects to have the conformal parafermionic field theory. Fateev and Zamolodchikov[3] have shown that at the points

$$x_k = \prod_{\ell=0}^{k-1} \frac{\sin(\frac{\pi \ell}{N} + \frac{\pi}{4N})}{\sin\left[\pi \frac{(\ell+1)}{N} - \frac{\pi}{4N}\right]} \qquad (5)$$

$$\tilde{x}_k = x_k$$

the star–triangle relation[4] is fulfilled, and so the continuun theory should be integrable too. As we have already mentioned, Fateev and Zamolodchikov introduce chiral fields (generalizing the Majorana fermion of the Ising model) in order to describe this continuum theory. So, before moving to the study of their theory let us see how the simplest of the chiral models can be constructed.

3. QUANTUM FIELD THEORY OF CHIRAL FIELDS

Another motivation to study chiral fields comes from the heterotic string theory[12]. Chiral fields also appear in the investigation of constrained field theories[13]. From this point of view Floreanini and Jackiw[16] have offered a beautiful solution to the problem of constructing the theory of a single chiral field. They have considered the Hamiltonian

$$H = \frac{1}{2} \int dx\, \psi^2(x) \qquad (6)$$

where x denotes space, and the unusual common time commutation relations

$$[\psi(x), \psi(y)] = i\delta'(x-y) \qquad (7)$$

which lead to the self–duality equation:

$$\dot{\psi} = i[H, \psi] = \psi' \qquad (8)$$

where the dot means differentiation with respect to time.

The associated classical (non–local) Lagrangian is

$$\mathcal{L} = \frac{1}{4}\int dx\, dy\, \psi(x)\, \epsilon(x-y)\, \dot{\psi}(y) - \frac{1}{2}\int dx\, \psi^2(y) \qquad (9)$$

where $\epsilon(x-y)$ denotes the step–function, whose Euler–Lagrange equations are

$$\psi(x) = \frac{1}{2}\int dy\, \epsilon(x-y)\, \dot{\psi}(y) \quad . \qquad (10)$$

In fact, the Lagrangian (9) describes a constrained dynamics. To see that it suffices to realize that the canonically conjugate momentum

$$\pi(x) = \frac{1}{4}\int dy\, \psi(y)\, \epsilon(y-x) \qquad (11)$$

is a constraint since it does not depend on the velocities. So,

$$T(x) = \pi(x) - \frac{1}{4}\int dy\, \psi(y)\, \epsilon(y-x) \simeq 0$$

$$\{T(x),T(y)\} = \frac{1}{2}\, \epsilon(x-y) \quad .$$

Girotti and Costa[7] have shown that this is the only (second class) constraint, and so one could employ the Dirac formalism for constrained systems, defining Dirac brackets by

$$\{f,g\}_D = \{f,g\} - \int dz\, dz'\{f,T(z)\}\, Q^{-1}(z,z')\, \{T(z'),g\}$$

where $Q(z,z') = \{T(z),T(z')\}$. Then one obtains

$$\{\psi(x),\psi(y)\}_D = \delta'(x-y)$$

which leads naturally to the equal time commutation relation (7). Also, the equation of motion for the chiral field ψ follows

$$\dot{\psi} = \{\psi,H\}_D = \psi' \quad .$$

The spin one chiral field ψ is interpreted as a charge density field, and so one might ask about the charge creating chiral fields that should be present in this model[8]. These fields are defined through their Dirac brackets with the field ψ by

$$\{\psi(x),u(y)\}_D = i\gamma\, \delta(x-y)\, u(y)$$

where γ is a free parameter. Thus, besides the chirality condition $\dot{u} = u'$, one gets the equation

$$u'(x) = -i\gamma\, \psi(x)\, u(x)$$

whose integral is

$$u(x) = e^{i\gamma\pi(x)} \quad . \qquad (12)$$

The field $u(x)$ has spin zero (as the field $\pi(x)$) in the classical case but adquires a dynamical spin (given in terms of γ) in the quantum regime. Indeed, the equation (12) is the source for the bosonization procedure in the quantum regime. As it will be explained in the end of the lecture the bosonization of the parafermionic fields is known but any classical counterpart (if it exists at all) is missing.

4. Z(N) PARAFERMIONIC MODELS

In order to make contact with the continuum theory, Fateev and Zamolodchikov[2] introduce one additional spin variable given by $\sigma_k(\vec{r}) = (\sigma(\vec{r}))^k$ $k = 1,2,\ldots,N-1$, which take value ω^{kq} and $\sigma_{N-k}(\vec{r}) = \sigma_k^*(\vec{r})$. These spin variables should correspond to the continuous conformal field $\sigma_k(y)$, $y \in \mathbb{R}^2$ ($\sigma_{N-k} = \sigma_k^\dagger$). Their dimensions are denoted by $2d_k$ such that $d_{N-k} = d_k$, and the Z(N) symmetry being defined by the invariance of the correlation functions under the substitution

$$\sigma_k(y) \longrightarrow \omega^{mk}\sigma_k(y) \qquad (13)$$

for $m \in \mathbb{Z}$. They also explain that the order–disorder duality implies that it should occur additional fields, $\mu_k(y)$ $k = 1,2,\ldots,N-1$, corresponding to the disorder parameters, with the same dimensions $2d_k$. The self–duality then implies that all correlation functions are invariant under the interchange $\sigma_k \leftrightarrow \mu_k$. (This means that the theory possesses an additional $\tilde{Z}(N)$ symmetry associated to the substitution corresponding to (13) for the fields $\mu_k(y)$. Thus the general fields in this model will be labeled by their $Z(N)$ and $\tilde{Z}(N)$ charges. These fields will not be discussed further in this lecture and the reader referred to the refs. [2,14] for a full account.)

In order to describe how the chiral fields arise it is convenient to introduce complex coordinates in $\mathbb{R}^2$ by $z = y_1+iy_2$ and $\bar{z} = y_1+y_2$. Reasoning then in analogy to the Ising model Fateev and Zamolodchikov postulate the operator product expansion (ope) of an order field and a disorder one as

$$\sigma_k(z,\bar{z})\mu_k(0,0) = z^{\Delta_k-2d_k}\,\bar{z}^{-2d_k}\,\psi_k(0) + \cdots \qquad (14)$$

where $\psi_k = \psi_k(z)$ are the chiral fields and Δ_k are their spins. (There are in addition left–handed chiral fields $\overline{\psi}_k = \overline{\psi}_k(\bar{z})$ but from now on we concentrate on right handed fields. Also we will suppress the $\bar{z}$ dependence of the magnetization fields to avoid repetition.) The field ψ_k is supposed to be conformal

$$T(z)\,\psi_k(z') = \frac{\Delta_z}{(z-z')^2}\,\psi_k(z') + \frac{1}{(z-z')}\,\partial_{z'}\,\psi_k(z') + \cdots \qquad (15)$$

where $T(z)$ is the chiral component of the energy momentum tensor and fulfills the Virasoro algebra

$$T(z)\,T(z') = \frac{c/2}{(z-z')^4} + \frac{2}{(z-z')^2}\,T(z) + \frac{1}{(z-z')}\,\partial_{z'}\,T(z') + \cdots \qquad (16)$$

The model is then fully specified by defining the algebra of the parafermion currents:

$$\psi_k(z)\,\psi_{k'}(z') = c_{k,k'}\,(z-z')^{\Delta_{k+k'}-\Delta_k-\Delta_{k'}}\,\psi_{k+k'}(z') + \cdots \qquad (17.a)$$
$$k + k' < N$$

$$\psi_k(z)\,\psi_{N-k}(z') = c_{k,N-k'}\,(z-z')^{\Delta_{k-k'}-\Delta_k-\Delta_{N-k'}}\,\psi_{k+k'}(z') \qquad (17.b)$$
$$k' < k$$

$$\psi_k(z)\,\psi_{N-k}(z') = (z-z')^{-2\Delta_k}\left[I + \frac{2\Delta_k}{c}\,(z-z')^2\,T(z') + \cdots\right] \qquad (17.c)$$

together with the input of choosing $\Delta_k = \frac{k(N-k)}{N}$. Then looking at the conformal Ward identities that follow from (15) and (16) one computes c to be

$$c = \frac{2(N-1)}{(N+2)}. \qquad (18)$$

Also, using (17) to decrease the order of an arbitrary 2n–point function of the fields $\psi_k(z)$ and demanding that the procedure gives the same result independently of the way that one fuses the fields $\psi_k(z)$, one gets

$$c^2_{k,k'} = \frac{\Gamma(k+k'+1)\;\Gamma(N-k+1)\;\Gamma(N-k'+1)}{\Gamma(k+1)\;\Gamma(k'+1)\;\Gamma(N-k-k'+1)\;\Gamma(N+1)}. \qquad (19)$$

At the end of the lecture we will compute the above structure constants using a different procedure. It will become clear then the group theory content of the numbers (19). Before this, we will present the definition of general parafermionic models[9].

5. GENERAL PARAFERMIONIC THEORY

We start recalling the definition of a level N Kac–Moody algebra g

$$J^a(z)\,J^b(z') = \frac{N\,g^{ab}}{(z-z')^2} + \frac{f^{abc}}{(z-z')}\,J(z') + \cdots \qquad (20)$$

where $J^a(z)$ stands for either the generators $E^{\alpha}(z)$ (α a root) or the Cartan subalgebra (CSA) generators $H^i(z)$, $i = 1,2,...,r$. The parafermionic theory is obtained by decoupling the CSA from g. To do that one defines r free–fields by

$$<\phi_i(z)\ \phi_j(z')> \ = \ -\ln(z-z')\ \delta_{ij} \qquad (21)$$

and writes the CSA generators as

$$H^j(z) \ = \ 2\sqrt{N}\ \frac{\alpha^j}{\alpha_j^2}\ \partial_z \phi \qquad (22)$$

where α_j is a simple root. (We will concentrate in the simply laced case. See ref. [9] for a full account.) The currents $E^{\alpha}(z)$ get then decomposed as

$$E^{\alpha}(z) \ = \ \sqrt{\frac{2N}{\alpha^2}}\ c_\alpha\ \psi_\alpha(z) : \exp(i\alpha\phi(z)/\sqrt{N}): \qquad (23)$$

where c_α is a cocycle factor and $\psi_\alpha(z)$ are parafermion fields. From (21) and (23) one sees that their common dimension is

$$\Delta_\alpha \ = \ 1 - \frac{\alpha^2}{2N} \ . \qquad (24)$$

Then it follows from (23) and the algebra (20) that the field $\psi_\alpha(z)$ must fulfill the parafermionic algebra:

$$\psi_\alpha(z)\ \psi_\beta(z') \ = \ k_{\alpha,\beta}\,(z-z')^{\Delta_{\alpha+\beta}-\Delta_\alpha-\Delta_\beta}\ \psi_{\alpha+\beta}(z') + \cdots \qquad (25)$$

if $\alpha+\beta$ is a root. $k_{\alpha,\beta}$ are certain numerical factor and in (25) we have used the convention that $\Delta_0 = 0$. To see how the remainder parafermionic fields appear, recall that we are dealing with integrable representations of Kac–Moody algebras[15]. So, there is a highest weight Λ which obeys $\Lambda \cdot \psi = N$ where ψ is the highest root of the algebra g (normalized to $\psi^2 = 2$). One verifies that the fields belonging to this representation get decomposed in a complete set of parafermionic fields times the CSA contribution. Indeed, Fuchs and Gepner[16] have observed that the four–point functions of these fields possess power–like behavior. Extracting the CSA factor one gets the parafermionic contribution. The details of this computation will be published elsewhere[17] and we will omit it here since it will cost us a long detour in the main theme of this introductory lecture.

Let us briefly see how to compute the central charge, C_P, of the parafermionic Virasoro algebra. Recall that the Virasoro generators for the WZW model[5] are given by

$$L_n^{WZW} \ = \ \frac{1}{(N+\tilde{h})} \sum_{m\in\mathbb{Z}} :J_n^a\, J_{n-m}^a:$$

where $\tilde{h}$ is the dual Coxeter number and $J^a(z) = \sum_{m\in\mathbb{Z}} \frac{J_m^a}{z^{m+1}}$. Denoting by L_n^{CSA} the Virasoro generators corresponding to the CSA and writting

$$L_n^{WZW} \ = \ L_n^{CSA} + L_n^P$$

one sees that C_p is given by

$$C_p \ = \ \frac{ND}{N+\tilde{h}} - r$$

where D is the dimension of the algebra g. Thus this construction is in fact a particular case of the Goddard–Kent–Olive coset construction[18].

6. COMPUTATION OF THE STRUCTURE CONSTANTS

We now turn to the Z(n) parafermionic field theory. Considering then the su(2) level N Kac–Moody algebra and writting

$$E(z) \ = \ \psi(z) \ = \ \exp\left[i\sqrt{\frac{2}{N}}\,\phi(z)\right]$$

for the positive root generator (and an analogous expression for the negative root one, F(z), the CSA being generated by $H = \sqrt{N}\,\partial_z\phi$). Consulting the work of

Bernard and Thierry–Mieg[19] one sees that $\psi(z)$ is represented by

$$\psi(z) = \sum_{\lambda \in \{\Lambda_1\}} \exp(i\sqrt{2}\,\lambda\cdot X(z))$$

where $\{\Lambda_1\}$ is the set of weights of the elementary representation of su(N) and the free fields $X^i(z)$ fulfill

$$<X^i(z)\, X^j(z')> \;=\; -\ln(z-z')\,\delta^{ij} \quad .$$

(We are now employing a different normalization from that used by Fateev and Zamolodchikov). Also, from the work of Fuchs and Gepner[16] it follows that the isospin $J = N/2$ field is given by

$$\Phi^J_J(z) = :\exp\left[i\sqrt{\frac{N}{2}}\,\phi(z)\right]: \quad .$$

Considering then the ope of $F(z)$ and a field in the multiplet $\Phi^J(z)$:

$$F(z)\,\Phi^J_j(z') = \frac{t_{jj-1}}{(z-z')}\,\Phi^J_{j-1}(z') + \cdots$$

one gets

$$\Phi^J_{J-j}(z) = \psi_{N-j}(z):\exp\left[i(N-2j)\,\frac{1}{\sqrt{2N}}\,\phi(z)\right]:$$

where

$$\psi_{N-j}(z) = \frac{j!}{\prod_{k=0}^{j-1} t_{J-kJ-k-1}} \sum_{\lambda \in \{\Lambda_{N-j}\}} :\exp(-i\sqrt{2}\,\lambda\cdot X(z)):$$

$\{\Lambda_k\}$ is the set of weight of the $\{k\}$ representation of $\partial\mu(N)$.

The $\psi_j(z)$ are the parafermion fields of Fateev and Zamolodchikov (with a different normalization). With their explicit expression the reader can easily perform their product obtaining in this way the structure constants (19). This computation and the analogous ones for other algebras will appear elsewhere[17].

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REFERENCES

1. Lepowski, J. and Wilson, G., Invent. Math. 77, 199 (1984).
2. Fateev, V.A. and Zamolodchikov, A.B., Sov. Phys. JETP 62, 215 (1985).
3. Fateev, V.A. and Zamolodchikov, A.B., Phys. Lett. 92A, 37 (1982).
4. Baxter, R.J., "Exactly Solved Models in Statistical Mechanics" (Academic Press, New York, 1982).
5. Knizhnik, V.G. and Zamolodchikov, A.B., Nucl. Phys. B247, 83 (1984); Gepner, D. and Witten, E., Nucl. Phys. B278, 493 (1986). Gomes, M., Kurak, V. and Silva, A.J. da, Nucl. Phys. B295, 139 (1988).
6. Floreanini, R. and Jackiw, R., Phys. Rev. Lett. 59, 1873 (1987).
7. Costa, M.E.V. and Girotti, H.O., Phys. Rev. Lett. 60, 1771 (1988).
8. Girotti, H., Gomes, M., Kurak, V., Rivelles, V.O. and Silva, A.J. da, Phys. Rev. Lett. 60, 1913 (1988).
9. Gepner D., Nucl. Phys. B290, 10 (1987).
10. Goddard, P. and Olive, D., Int. J. Mod. Phys. A1, 303 (1986).
11. Alcaraz, F.C. and Koberle, R., J. Phys. A13, L153 (1980).
12. Green, M., Schwarz, J. and Witten, E., "Superstring Theory" (Cambridge Univ. Press, Cambridge, 1987).
13. Siegel, W., Nucl. Phys. B238, 307 (1984).
14. Gepner, D. and Qiu, Z., Nucl. Phys. B285, 423 (1987).
15. Kac, V.G. and Peterson, D., Adv. Math. 53, 125 (1984).
16. Fuchs, J. and Gepner, D., Nucl. Phys. B294, 30 (1987).
17. Kurak, V., to be published.
18. Goddard, P., Kent, A. and Olive, D., Comm. Math. Phys. 103, 105 (1986).
19. Bernard, D. and Thierry–Mieg, J., Comm, Math. Phys. 111, 181 (1987).