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ABSTRACT: A new system of coordinates was found and a method was developed to determine a toroidal equilibrium of plasmas with an arbitrary current distribution and arbitrary plasma cross–section. The method depends on the knowledge of the equilibrium of a straight plasma column of similar cross section and similar current distribution. Large aspect–ratio is assumed. By successive approximations better solutions can be obtained. An explicit formula for the poloidal flux of a nearly circular plasma is presented. This can be written in a very suggestive form, in terms of a function related to the asymmetry of the poloidal field due to toroidality. The method works provided there is only one magnetic axis.

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1. INTRODUCTION

MHD equilibrium in axisymmetric tokamaks can be described by the poloidal flux Ψ which satisfies an elliptic partial differential equation (Shafranov 1966):

$$R^2\nabla \cdot (R^{-2}\nabla\Psi) = \mu_0 J_3 = \mu_0 R J_\phi = -R^2\mu_0 \frac{dP}{d\Psi} - \mu_0^2 I \frac{dI}{d\Psi} \qquad (1)$$

Here R is the major radius and J_ϕ the toroidal current density. The plasma pressure P and the poloidal current I are surface quantities, thus, depending on position through Ψ only.

Equilibrium distribution of the plasma current in a tokamak cannot be totally arbitrary (Coppi 1980; Brusati et al 1984; Biskamp 1986; Pfirsch and Pohl 1988). The question of profile consistency in tokamaks is still unsolved. However, some relevant results have been reported in this context: Egorov et al (1987) developed a method to measure the direction of the magnetic field in T–10 tokamak which can be used to calculate the profile of the safety factor q. From this result, current distribution can be inferred. Kuznetsov et al (1987) observed that magnetic measurements external to a tokamak plasma can not only reveal some global properties, such as the total current and the mean kinetic presure, but it can be used to find the current distribution, provided restrictions are made as to the class of possible current profiles.

The profiles of toroidal current J_ϕ and $\frac{dP}{d\Psi}$ are expected to be very similar (Kuznetsov et al. 1986). It seems that, in general, the current distribution is not a linear function of Ψ. Thus, the self–consistent Grad–Shafranov equation is a nonlinear elliptic equation containing a scarcely known function of Ψ. It seems that in tokamak plasmas conditions the solution is unique (Field and Papaloizon 1977) if Dirichlet's boundary condition is assumed.

Yoshikawa (1974) developed a method to determine a toroidal equilibrium of plasmas using local polar coordinates once a cylindrically symmetric linear solution is known. Arbitrary current distribution can be treated by this method. Hazeltine and Montgomery (1988) adopted the same approach to find the equilibrium of a nearly circular low β plasma using a different coordinates system.

We found a curious system of coordinates to describe toroidal equilibria of plasmas with an arbitrary current distribution and arbitrary cross section. In the limit of large aspect–ratio it becomes the standard local polar coordinates system (§2).

The idea of bending a straight cylindrical plasma into a torus is taken from Yoshikawa's work. The modification in the equilibrium due to toroidality is described by an almost linear elliptic equation. The nonlinear term is a quantity of the order of the inverse aspect–ratio.

In axially symmetric systems with large aspect–ratio, the correction can be treated as a perturbation and the equation solved by successive approximations. Thus, results known for the case of straight cylinder of any cross section can be taken over to toroidal symmetry (§3).

A fundamental assumption in this work and the former ones (Yoshikawa 1974; Hazeltine and Montgomery 1988) is that J_ϕ has the same functional dependence on Ψ as in the corresponding straight cylindrical plasma.

In §4 the equation (1) is solved for the case of nearly circular plasma, correct up to first order in inverse aspect–ratio. Bending straight circular cylinder of radius r in a torus results in a nearly elliptical toroid with eccentricity $\simeq r/2R$. Dirichlet boundary condition is applied on the outermost elliptical toroid with eccentricity $\simeq a/2R$. This corresponds to taking a fixed plasma boundary. The poloidal flux can be written in a very suggestive form as a sum of two terms. The first term is represented by a cylindrical plasma function which contains much of toroidal effect through the coordinate that is used. The second term, which is of the order of a/R_0, is written in terms of a function related to

the asymmetry of the poloidal field in the equatorial plane. A curious result is that for low β_p the shape of the magnetic surfaces are almost model independent. The asymmetry of poloidal field in the equatorial plane depends on β_p and on the current distribution.

The model works provided the poloidal field does not vanish inside the plasma.

2. TOROIDAL POLAR COORDINATES

The usual toroidal coordinates system as used by Shafranov (1966) and denoted here by (ξ, ω, φ) are defined in terms of circular cylindrical coordinates (R, φ, Z) by:

$$R = \frac{R_0' \sinh\xi}{\cosh\xi - \cos\omega} \quad \text{and} \quad Z = \frac{R_0' \sin\omega}{\cosh\xi - \cos\omega} \tag{2}$$

where R_0' is related to the geometric major axis R_0 of the plasma by:

$$R_0' = R_0 \sqrt{1 - a^2/R_0^2}$$

A new set of coordinates (ρ_t, θ_t, φ) is defined here by:

$$\rho_t = R_0' / (\cosh\xi - \cos\omega)$$

and (3)

$$\theta_t = \pi - \omega$$

The meaning of the coordinates ρ_t and θ_t can be perceived by observing the figure 1.

Relations between this and local polar system (ρ, θ, φ) with the same origin is:

$$\rho_t = \rho\left(1 - \frac{\rho}{R_0'}\cos\theta + \left(\frac{\rho}{2R_0}\right)^2\right)^{1/2}$$

and (4)

$$\sin\theta_t = \sin\theta \left(1 - \frac{\rho}{R_0'}\cos\theta + \left(\frac{\rho}{2R_0}\right)^2\right)^{-1/2}$$

These clearly show that in the limit $\rho/R_0' \ll 1$ ρ_t and θ_t become ρ and θ, respectively.

ρ_t = constant is nearly elliptic surface.

3. EQUILIBRIUM EQUATION

The equilibrium equation (1) in terms of the toroidal polar coordinates becomes (Appendix):

$$\frac{1}{\rho_t}\frac{\partial}{\partial\rho_t}\rho_t\frac{\partial\Psi}{\partial\rho_t}+\frac{1}{\rho_t^2}\frac{\partial^2\Psi}{\partial\theta_t^2}=\mu_0 J_{30}(\Psi)+\mu_0 R_0'^2\frac{dP}{d\Psi}\left(2\frac{\rho_t}{R_0'}\cos\theta_t+\frac{\rho_t^2}{R_0^2}\sin^2\theta_t\right)$$
$$+\frac{\rho_t\cos\theta_t}{R_0'}\left(2\frac{\partial^2\Psi}{\partial\rho_t^2}+\frac{1}{\rho_t}\frac{\partial\Psi}{\partial\rho_t}\right)+\frac{\rho_t\sin^2\theta_t}{R_0'}\left(\frac{1}{\rho_t}\frac{\partial\Psi}{\partial\theta_t}-\frac{2}{\rho_t}\frac{\partial^2\Psi}{\partial\theta_t\partial\rho_t}\right) \qquad (5)$$

where

$$\mu_0 J_{30}(\Psi)=-\mu_0 R_0'^2\frac{dP}{d\Psi}-\frac{d}{d\Psi}\frac{\mu_0^2 I^2}{2} \qquad (6)$$

The contravariant components of poloidal field are given by:

$$\vec{B}\cdot\nabla\theta_t=\frac{1}{R_0'\rho_t}\frac{\partial\Psi}{\partial\rho_t}$$
$$\vec{B}\cdot\nabla\rho_t=-\frac{1}{R_0'\rho_t}\frac{\partial\Psi}{\partial\theta_t} \qquad (7)$$

and the toroidal field by:

$$B_3=RB_\phi=-\mu_0 I \qquad (7')$$

In the limit of large aspect-ratio, the zero-th order equation is taken as:

$$\frac{1}{\rho_t}\frac{\partial}{\partial\rho_t}\rho_t\frac{\partial\Psi_0}{\partial\rho_t}+\frac{1}{\rho_t^2}\frac{\partial^2\Psi_0}{\partial\theta_t^2}=\mu_0 J_{30}(\Psi_0) \qquad (8)$$

This expression is identical to the Grad–Shafranov equation for a straight cylindrical plasma with an arbitrary cross section except for the definition of the coordinates. In both cases Ψ_0 does not depend on a third coordinate. Once the cylindrical solution $\Psi_0(\rho,\theta)$ is

known, the zero–th order solution of the toroidal system will be given by $\Psi_0(\rho_t,\theta_t)$.

The equilibrium solution is, then, written as:

$$\Psi(\rho_t,\theta_t)=\Psi_0(\rho_t,\theta_t)+\delta\Psi(\rho_t,\theta_t) \qquad (9)$$

Here we assume that the current terms can be very nearly represented by:

$$J_{30}(\Psi)\simeq J_{30}(\Psi_0)+\left(\frac{d}{d\Psi_0}J_{30}(\Psi_0)\right)\delta\Psi$$

and

$$\frac{dP}{d\Psi}(\Psi)\simeq\frac{d}{d\Psi_0}P(\Psi_0)+\left(\frac{d^2}{d\Psi_0^2}P(\Psi_0)\right)\delta\Psi \qquad (10)$$

The pressure profile in the cylindrical approximation is taken as:

$$-R_0'^2\frac{dP}{d\Psi_0}(\Psi_0)=\beta_p J_{30}(\Psi_0) \qquad (11)$$

where β_p is the poloidal beta defined as the ratio between the average kinetic pressure and the magnetic pressure at the plasma surface due to the poloidal field.

With these assumptions and observing that:

$$\nabla(\nabla^2\Psi_0)=\mu_0\left(\frac{d}{d\Psi_0}J_{30}(\Psi_0)\right)\nabla\Psi_0 \qquad (12)$$

the equilibrium equation for $\delta\Psi$ can be deduced from (5) as:

$$(\nabla_t\Psi_0)\,\nabla_t^2\delta\Psi-(\nabla_t(\nabla_t^2\Psi_0))\,\delta\Psi=\frac{\rho_t}{R_0}\chi\,\nabla_t\Psi_0 \qquad (13)$$

∇_t and ∇_t^2 are the cylindrical gradient and laplacian operators relative to the coordinates ρ_t and θ_t:

$$\nabla_t\equiv(\nabla\rho_t)\frac{\partial}{\partial\rho_t}+\nabla\theta_t\frac{\partial}{\partial\theta_t}$$
$$\nabla_t^2\equiv\frac{1}{\rho_t}\frac{\partial}{\partial\rho_t}\rho_t\frac{\partial}{\partial\rho_t}+\frac{1}{\rho_t^2}\frac{\partial^2}{\partial\theta_t^2} \qquad (14)$$

and

$$\chi\equiv\cos\theta_t\left[2\frac{\partial^2\Psi}{\partial\rho_t^2}+\frac{1}{\rho_t}\frac{\partial\Psi}{\partial\rho_t}-2\beta_p\mu_0 J_{30}(\Psi)\right]+\sin\theta_t\left(\frac{1}{\rho_t^2}\frac{\partial\Psi}{\partial\theta_t}-\frac{1}{\rho_t}\frac{\partial^2\Psi}{\partial\rho_t\partial\theta_t}\right)$$
$$-\frac{\rho_t}{R_0}\sin^2\theta_t\,\beta_p\mu_0 J_{30}(\Psi) \qquad (15)$$

Defining a function $\phi_0 \equiv \vec{C} \cdot \nabla_t \Psi_0$ where $\vec{C}$ is an arbitrary constant vector :

$$\phi_0(\nabla_t^2\, \delta\Psi) - (\nabla_t^2 \phi_0)\, \delta\Psi = \phi_0 \frac{\rho_t}{R_0}\, \chi \tag{16}$$

Writting $\delta\Psi \equiv \frac{a}{R_0} \phi_0\, f$ we get:

$$\nabla_t \cdot ((\nabla_t f)\, \phi_0^2) = \frac{\rho_t}{a} \phi_0\, \chi \tag{17}$$

Here, χ depends on Ψ. The 1st order approximation is obtained substituting Ψ by Ψ_0 in the 2nd member of the equation (17). The solution of the eq. (5) will be:

$$\Psi \simeq \Psi_0 + \frac{a}{R_0} \phi_0\, f_1 \tag{18}$$

where f_1 is the solution of the nonhomogeneous linear elliptic equation (17), the nonhomogeneous term will be a known function of position once the cylindrical solution is determined; this correction is of the order of the inverse aspect–ratio.

Equation (17) can be dealt as a partial differential equation in conventional cylindrical coordinates. In order to solve this equation $\vec{C}$ can be chosen as the unit vector in the direction $\theta_t = 0$ so that $\phi_0 = \frac{\partial \Psi_0}{\partial \rho_t} \cos\theta_t$.

For a plasma with fixed boundary the outermost magnetic surface can be determined by $\Psi_0\, (\rho_t, \theta_t) =$ constant and $\delta\Psi = 0$. The solution is expected to be unique under tokamak conditions (Field and Papaloizon 1977).

Higher accuracy can be reached by successive approximation method. The original problem is significantly simplified by this method.

4. APPLICATION TO A TOROIDAL PLASMA WITH NEARLY CIRCULAR CROSS SECTION

0th order solution is assumed to be an axially symmetric solution of the equation:

$$\frac{1}{\rho_t} \frac{d}{d\rho_t} \rho_t \frac{d\Psi_0}{d\rho_t} = \mu_0 J_{30}(\Psi_0) = \mu_0 R_0' J_\phi \tag{19}$$

Taking $\phi_0 = \Psi_0' \cos\theta_t$ and considering terms up to the order of ρ/R_0:

$$\nabla \cdot (\nabla f \cos^2\theta_t\, \Psi_0'^2) \simeq \frac{\cos\theta_t}{a} ((1 - \beta_p) \frac{1}{\rho} \frac{d}{d\rho} (\rho \Psi_0')^2 - \Psi_0'^2) \tag{20}$$

Here, the primo is the derivative with respect to ρ_t.

There is a solution for f depending on ρ_t only:

$$f' \simeq \frac{1 - \beta_\rho}{a} \rho_t - \frac{1}{\rho_t \Psi_0'^2} \int_0^{\rho_t} \frac{\rho}{a} \Psi_0'^2\, d\rho \tag{21}$$

Defining $\Lambda(\rho_t)$:

$$\Lambda(\rho_t) = -\frac{a}{\rho_t} \frac{df}{d\rho_t} = 1 - \beta_p - \frac{1}{\rho_t^2\, \Psi_0'^2} \int_0^{\rho_t} \rho\, \Psi_0'^2\, d\rho \tag{22}$$

the poloidal flux becomes:

$$\Psi \simeq \Psi_0\, (\rho_t) + \Psi_0'(\rho_t) \cos\theta_t \int_{\rho_t}^{a} \frac{\rho}{R_0} \Lambda(\rho)\, d\rho \tag{23}$$

It is curious to notice that $\Lambda(a)$ is the coefficient of asymmetry of the poloidal field defined in terms of poloidal β_p and the normalized internal inductance per unit lenght (see for e.g. Mukhovatov and Shafranov 1971):

$$\Lambda(a) = \frac{l_i}{2} + \beta_p - 1$$
$$l_i = \frac{2}{B_\theta^2\, a^2} \int_0^a B_\theta^2\, \rho d\rho \tag{24}$$

$\rho_t = a$ is the outermost plasma surface.

Other choices of $\vec{C}$ must lead to the same result once the boundary conditions are established.

Figure 2 shows the surfaces $\rho_t =$ constant and circles around the origin corresponding to local polar coordinate $\rho =$ constant.

The poloidal magnetic field is calculated from (7). The toroidal magnetic field is deduced from (7') as:

$$RB_\phi \simeq -I(\Psi_0) + \frac{1}{I(\Psi_0)} (1 - \beta_p)\, J_{30}(\Psi_0)\, \delta\Psi \tag{25}$$

In the outermost surface the poloidal field is very nearly given by:

$$B_p \simeq \frac{\Psi_0'(a)}{R_0}\left(1 - \frac{a}{R_0'} \Lambda(a) \cos\theta_t\right) \qquad (26)$$

Poloidal flux and magnetic field in the equatorial plane were calculated numerically assuming a peaked current density:

$$J_{30} = \text{constant } \left(1 - \left(\frac{\rho_t}{a}\right)^2\right)^3 \qquad (27)$$

and using typical values for parameters found in TBR–1 tokamak (Vannucci et al 1989):

plasma current	$I_p = 10$ kA
major radius	$R_0 = 0.30$ m
radius of limitator	$a = 0.08$ m
toroidal field	$B_\phi = 0.5$ T
	$\Lambda(a) = 0.28$

The first order correction is very small. 0^{th} order and 1^{st} order solutions cannot be distinguished in fig. 2.

Figure 3 shows the poloidal field in 0^{th} order and 1^{st} order approximations. It seems that the magnetic surfaces for low β_p and nearly circular cross section plasmas are almost model independent. The 1^{st} order correction accounts for the asymmetry of poloidal field on the magnetic surface. This term depends on β_p and the current profile.

5. CONCLUSION

Following Yoshikawa, a method was developed to solve a self consistent equilibrium equation with toroidal geometry using a new set of coordinates. The major problem is reduced to searching the cylindrical solution with more realistic current distribution and cross–section. Once this solution is found the method developed here allows us to find the departure from the cylindrical solution due to toroidality. The method is simple although some mathematical developments have to be performed.

In the case of nearly circular cross section plasma the 0^{th} order solution

contains much of the toroidal effect. For low β_p this term is weakly dependent on current profile. Also, the shape of the magnetic surfaces depends very little on current distribution. The next order term takes account of the asymmetry of the poloidal field on the same magnetic surface.

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APPENDIX

EQUILIBRIUM EQUATION IN TOROIDAL POLAR COORDINATES

Coordinates

$$x^1 \equiv \rho_t = \frac{R_0'}{\cosh\xi + \cos\theta_t}$$

$$x^2 \equiv \theta_t = \pi - \omega$$

$$x^3 \equiv \varphi$$

Contravariant Basis

$$e^1 = \nabla\rho_t = -\rho_t \frac{\sinh\xi}{R_0'}\vec{e}_\xi + \rho_t \frac{\sin\theta_t}{R_0'}\vec{e}_\theta$$

$$e^2\, \nabla\theta_t = \frac{\vec{e}_{\theta_t}}{\rho_t}$$

$$e^3 = \nabla\varphi = \frac{\vec{e}_\varphi}{R}$$

Contravariant Matrix Elements

$$g^{11} = 1 - \frac{2\rho_t}{R_0'}\cos\theta_t \qquad g^{12} = \frac{\sin\theta_t}{R_0'}$$

$$g^{22} = \frac{1}{\rho_t^2} \qquad g^{33} = \frac{1}{R^2}$$

$$\sqrt{g} = R_0'\,\rho_t \qquad R = R_0'\left(1 - \frac{2\rho_t}{R_0'}\cos\theta_t + \frac{\rho_t^2}{R_0'^2}\sin^2\theta_t\right)^{1/2}$$

Equilibrium Equation:

$$R^2 \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,\frac{1}{R^2}\,g^{ij}\frac{\partial\Psi}{\partial x^j}\right) = -\mu_0 R^2 \frac{dP}{d\Psi} - \mu_0^2 I \frac{dI}{d\Psi} = -R_0^2 \mu_0 \frac{dP}{d\Psi} - \mu_0^2 I \frac{dI}{d\Psi}$$

$$+ \mu_0 R_0^2 \frac{\rho_t}{R_0}\left(2\cos\theta_t + \frac{\rho_t}{R_0}\sin^2\theta_t\right)\frac{d\rho}{d\psi}$$

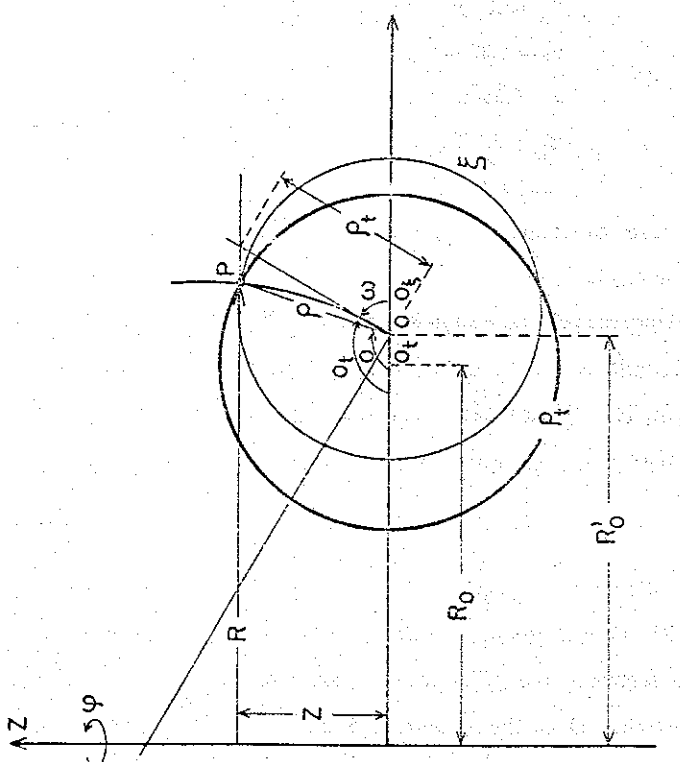

Figure 1 – Coordinates systems. O is the common origin of the coordinates systems. O_ξ is the geometric axis of constant ξ toroid and O_t is the geometric axis of constant ρ_t toroid.

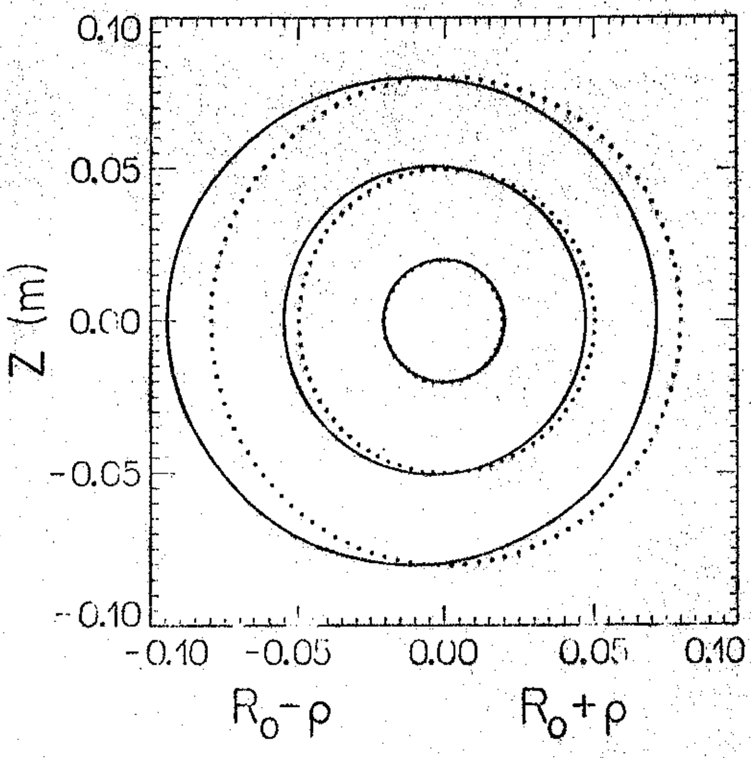

Figure 2 – Magnetic surfaces. Solid lines are the magnetic surfaces, coincident with constant ρ_t surfaces for TBR–1 tokamak. The points are constant ρ curves.

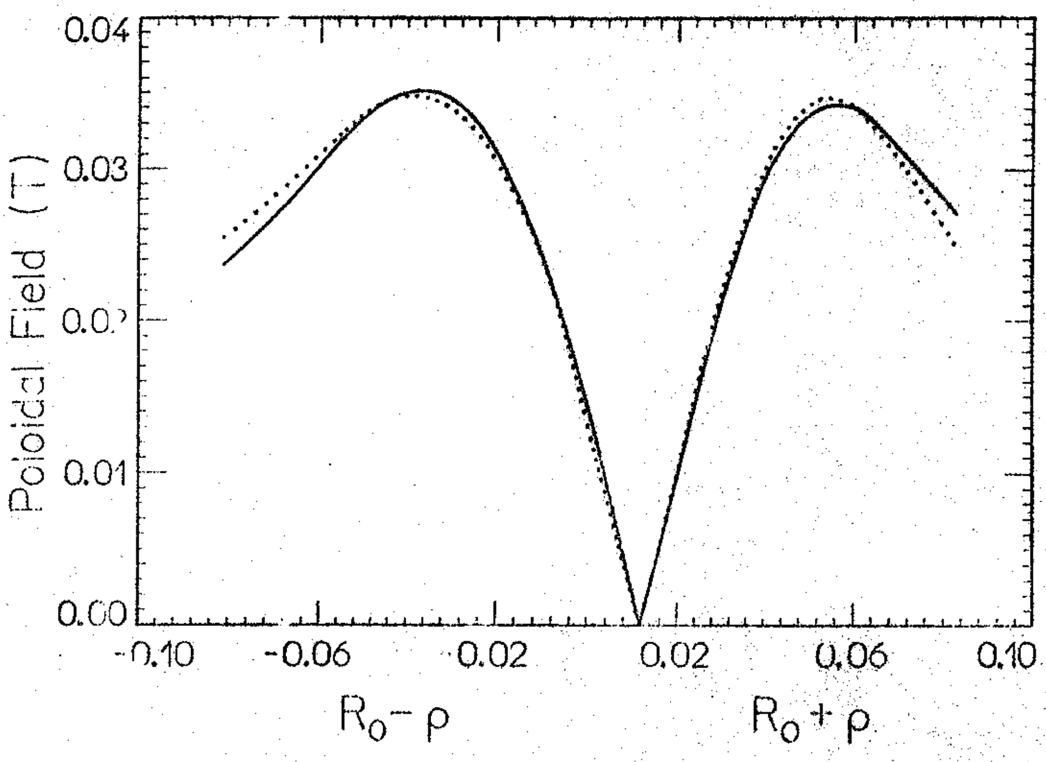

Figure 3 – The solid lines represent the poloidal field in 1^{st} order approximation. The points correspond to 0^{th} order approximation.