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Abstract

Two methods for selecting collective bosons, one proposed by Klein and Vallières and the other one being a number conserved Tamm Dancoff method, are applied in this work to boson mapping methods. The first mapping to be tested is a Dyson boson mapping in the SD shell and the second one is a mapping developed by Bonatsos, Klein and Li and applied to two j-shells with $|j_1 - j_2| = 4$. Whenever the boson mappings are accurate, the selection of collective bosons gives good results, independently of the method considered.

1.Introduction

Boson mappings have been used to establish a link between the interacting boson model (IBM) [1] and the shell model and hence, justify microscopically the success of the IBM. Recently the accuracy of some boson mappings has been tested [2]. It is well known that some methods have been successful so far in describing vibrational nuclei when tested in single j-shells though it is still not clear whether there is any good mapping to describe rotational nuclei. More realistic mappings which take into account many non-degenerate j-shells may be more appropriate for describing a larger range of nuclei.

In a single j-shell there is always just one kind of s-boson, one kind of d-boson, etc, independently of the mapping considered. When non-degenerate j-shells are examined, more than one kind of s-boson, one kind of d-boson, etc are necessary. Because of this fact, collective bosons which are responsible for some of the low-lying states must be selected in order to get the IBM type interactions.

Here we consider in some detail two methods for selecting collective bosons. The first one was proposed by Klein and Vallières [3] and the second one is a number conserved Tamm Dancoff method. To see how well these methods work to choose the collective bosons, we apply them to simple mapping cases.

The first test is performed in the sd shell where a Dyson boson mapping developed by Bonatsos and Klein [4] is used. In this method the bosons are generated by different irreducible representations of SU(3) which allow us to use the Elliott's SU(3) model [5] to calculate energy levels. The collective bosons are selected via Klein and Vallières method.

For the second test we consider BKL's boson mapping [6] and describe two j-shells with $|j_1 - j_2| = 4$ [7]. We use both above cited methods for choosing collective bosons.

Next section we show the calculations and results for both approaches.

2.Calculations and Results

2.1.Studying Dyson Mapping in the sd Shell

According to Bonatsos and Klein [4], the boson image of a fermion quadrupole operator in the sd shell which obeys SU(3) commutation relations is given by

$$
\begin{aligned}
B^2_\mu(SU(3)) = \\
\sqrt{\frac{14}{5}}(a_0^\dagger \tilde{a}(2)_\mu + a^\dagger(2)_\mu a_0) + (s^\dagger \tilde{d} + d^\dagger s) \\
-\frac{11}{2\sqrt{7}}[a^\dagger(2) \otimes \tilde{a}(2)]^2_\mu + \frac{\sqrt{7}}{2}[d^\dagger \otimes \tilde{d}]^2_\mu \\
+9\sqrt{\frac{2}{35}}([a^\dagger(4) \otimes \tilde{a}(2)]^2_\mu + [a^\dagger(2) \otimes \tilde{a}(4)]^2_\mu) \\
-3\sqrt{\frac{11}{14}}[a^\dagger(4) \otimes \tilde{a}(4)]^2_\mu,
\end{aligned}
\tag{2.1.1}
$$

where $a_0^\dagger$, $a^\dagger(2)_\mu$ and $a^\dagger(4)_\mu$ are boson creation operators belonging to the (4,0) irreducible representation (irrep) of SU(3) with angular momentum 0, 2, 4 respectively and $s^\dagger$ and $d^\dagger_\mu$ are boson creation operators belonging to the (0,2) irrep of SU(3) with angular momentum 0, 2 respectively . The boson annihilation operators belong to the corresponding irreps of SU(3). These irreducible representations are obtained with the consideration that in the sd shell each fermion has (2,0) SU(3) quanta and thus, (4,0) and (0,2) are the symmetric representations for two particle states. Boson operators within each irrep obey standard boson commutation relations while boson operators belonging to different SU(3) irreps commute.

We now consider a simple quadrupole Hamiltonian, which can be compared with the usual Elliott's Hamiltonian [5)] in the fermion space and is given in terms of the Dyson mapped operator in the boson space. The quadrupole Hamiltonians are

$$H_{Elliott} = -Q.Q = -\frac{1}{2}(\lambda^2 + \mu^2 + 3\lambda + 3\mu + \lambda\mu) + \frac{3}{8}L(L+1) \tag{2.1.2}$$

where λ and μ are the indices of the irreducible representation of SU(3) (λ, μ) and

$$H_{Dyson} = -B^2(SU(3)) \cdot B^2(SU(3)). \tag{2.1.3}$$

We diagonalize both Hamiltonians for n=1 boson (2 fermions) and obtain the results shown in table I.

Noticing that the results given by the mapped Hamiltonian are exact as expected, we turn our attention to the selection of collective bosons. According to Klein and Vallières (hereafter this method is denoted as KV) , the trace of the Hamiltonian does not depend on the basis which is used for its diagonalization and the condition determining the collective degrees of freedom is,

$$\delta(TrH_D) = 0 \tag{2.1.4}$$

where H_D contains only the diagonal terms of the Hamiltonian.

Rewriting the boson operators appearing in (2.1.1) in terms of just one collective s-boson and one collective d-boson, they read

$$a_0^\dagger = \alpha_1 S^\dagger \ , \ s^\dagger = \alpha_2 S^\dagger \tag{2.1.5}$$

$$a^\dagger(2) = \beta_1 D^\dagger \ , \ d^\dagger = \beta_2 D^\dagger \tag{2.1.6}$$

and $a^\dagger(4)$ remains the same, where

$$\alpha_1^2 + \alpha_2^2 = 1 \ , \ \beta_1^2 + \beta_2^2 = 1, \tag{2.1.7}$$

are necessary as normalization conditions. Substituting (2.1.5) and (2.1.6) into (2.1.1), rewriting (2.1.3) in terms of the new boson operators (in normal order) and calculating the trace in (2.1.4), one obtains

$$Tr(H_D) = -2\times(14\alpha_1^2 + 5\alpha_2^2 + 11.75\beta_1^2 + 2.75\beta_2^2 + 4.32\beta_1^4 + 1.75\beta_2^4 - 2.75\beta_1^2\beta_2^2 + 10.43). \tag{2.1.8}$$

Normal ordering is very important when utilizing the KV method to select collective bosons because originally, every boson should have been written in terms of a collective and a non-collective part. Here we drop the non-collective part of each boson and then, just normal ordered Hamiltonians give the correct result. Before continuing our calculation, it is worth pointing out that for the full Hamiltonian H_{Dyson}, the states with L=0 are $|a_0\ a_0>$, $|s\ s>$, $|a_0\ s>$, $|a(2)\ a(2)>$, $|d\ d>$, $|a(2)\ d>$, $|a(4)\ a(4)>$ for two-boson systems. When using collective bosons, we are restricting ourselves to the states $|S\ S>$, $|D\ D>$ and $|a(4)\ a(4)>$. The same kind of restriction is imposed for two boson systems with L=1,2,3. Defining

$$F(\alpha_i, n) = \frac{1}{n} Tr(H_D) \tag{2.1.9}$$

where i=1,2,3,4 and n is the number of bosons (here it is 2) and setting

$$\alpha_1 = sin\theta \ , \ \alpha_2 = cos\theta,$$

$$\beta_1 = sin\phi \ , \ \beta_2 = cos\phi \tag{2.1.10}$$

we obtain

$$F(\theta, \phi, 2) = -14sin^2\theta - 5cos^2\theta - 11.75sin^2\phi - 2.75cos^2\phi$$

$$-4.32sin^4\phi - 1.75cos^4\phi - 2.75sin^2\phi cos^2\phi - 10.43. \tag{2.1.11}$$

Calculating $\frac{\partial F}{\partial \theta} = 0$ and $\frac{\partial F}{\partial \phi} = 0$, we find out that either $sin\theta = 1$ and $cos\theta = 0$ or $sin\theta = 0$ and $cos\theta = 1$ and either $sin\phi = 1$ and $cos\phi = 0$ or $sin\phi = 0$ and $cos\phi = 1$. We have confirmed that the absolute minimum of the trace is given by (except for arbitrary signs)

$$\alpha_1 = \beta_1 = 1 \ , \ \alpha_2 = \beta_2 = 0 \tag{2.1.12}$$

which actually is the same as restricting the calculation to the (4,0) irrep of SU(3). Therefore the (4,0) representation is responsible for contributing with the low-lying part of the spectrum. In table II we write explicitly the states in the SD shell for 4 fermions and for 2 bosons. Considering spin and isospin degrees of freedom, all representations shown in table II for the fermion space are allowed but, since we consider only spin degrees of freedom, those representations belonging to [4] and [31] are not allowed. In this case, all states belonging to (80) and one state belonging to (42) irreps in the boson space are spurious as one can see in table II.

The results obtained for the full quadrupole Hamiltonian diagonalization and for the *collective Hamiltonian* diagonalization when n=2 bosons are given in table III and compared with some of the results obtained by Elliott's formula (2.1.2). As expected Dyson mappings give exact results except for those spurious states and the results in the KV are those coming from the (40) reps. only.

2.2. Studying BKL [7] for $|j_1 - j_2| = 4$

Next we consider a BKL mapping for two non-degenerate j-shells with $|j_1 - j_2| = 4$ and investigate the collective contributions of its operators. We can justify this choice by looking at the major shell 82-126 which can be regarded as two levels, namely $j_1 = 17/2$ and $j_2 = 25/2$, if we think of them in terms of occupation numbers: levels $h_{9/2}$ and $f_{7/2}$ can accommodate 18 fermions and are substituted by $j_1 = 17/2$ and levels $p_{1/2}$, $p_{3/2}$, $f_{5/2}$ and $i_{13/2}$ can accommodate 26 fermions and are substituted by $j_2 = 25/2$. Levels 17/2 and 25/2 can couple to J=4 but cannot couple to J=1,2 or 3. This fact facilitates our study because it reduces the number of considered bosons. In the BKL we have two fermions in level 1 (or 2) forming a pair with J=0 or J=2 which corresponds to an s- or d-boson, respectively.

We now consider a pairing plus quadrupole Hamiltonian

$$H = -xH_P - (1-x)H_{Q.Q} \tag{2.2.1}$$

and we diagonalize this hamiltonian for the exact case (in the fermion space), for the BKL mapping and for the collective KV case in which only collective bosons selected by Klein's method are included. We vary x from 0.0 to 1.0 in order to simulate from deformed to spherical nuclei. We can perform the diagonalization procedure only up to four-fermion systems due to computational limitation.

We start by looking at a pairing Hamiltonian in the fermion space given by

$$H_P = -(\sqrt{\Omega_1}A_0^\dagger(11) + \sqrt{\Omega_2}A_0^\dagger(22))\cdot(\sqrt{\Omega_1}A_0(11) + \sqrt{\Omega_2}A_0(22)) \tag{2.2.2}$$

where the J=0 pair creation operator is

$$A_0^\dagger(11)^F = \frac{1}{\sqrt{2}}\sum \frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}} a^\dagger_{m_1} a^\dagger_{-m_1} \tag{2.2.3}$$

and its exact boson image is

$$A_0^\dagger(11)^B = a^\dagger(1)\sqrt{r_1} \tag{2.2.4}$$

where $a^\dagger(1)$ is a boson creation operator with angular momentum 0 and

$$r_1 = 1 - \frac{n_1 + 2n_{11}}{\Omega_1}. \tag{2.2.5}$$

Here n_1 stands for the number of a(1)-bosons, n_{11} for the number of a(11)-bosons (which has angular momentum 2) and Ω_1 is the degeneracy of level 1. For level 2 we have the same equations just by exchanging indices 1 and 2.

The pairing Hamiltonian (2.2.2) in terms of boson operators reads

$$H_P(B) = -(a^\dagger(1)\sqrt{\Omega_1 r_1} + a^\dagger(2)\sqrt{\Omega_2 r_2})\cdot(\sqrt{\Omega_1 r_1}a(1) + \sqrt{\Omega_2 r_2}a(2)) \tag{2.2.6}$$

Now we look for collective bosons via Klein's method. In the following we write the original bosons in terms of the collective ones. For simplicity we consider only $a(1)$ and $a(2)$ bosons. They are expressed by the collective boson as

$$a^\dagger(1) = \alpha_1\, s^\dagger \;,\;\; a^\dagger(2) = \alpha_2\, s^\dagger \tag{2.2.7}$$

where we have the normalization condition:

$$\alpha_1^2 + \alpha_2^2 = 1. \tag{2.2.8}$$

The natural choice for α_1 and α_2 is

$$\alpha_1 = sin\theta \;,\;\; \alpha_2 = cos\theta. \tag{2.2.9}$$

Substituting (2.2.9) into (2.2.6) and calculating the trace, we find

$$F(\theta,1) = Tr(H_P(B)) = -(\Omega_1 sin^2\theta + \Omega_2 cos^2\theta + 2\sqrt{\Omega_1\Omega_2}sin\theta cos\theta). \tag{2.2.10}$$

Minimizing $F(\theta,1)$ we obtain

$$\alpha_1 = \sqrt{\frac{\Omega_1}{\Omega}} \;,\;\; \alpha_2 = \sqrt{\frac{\Omega_2}{\Omega}} \tag{2.2.11}$$

where $\Omega = \Omega_1 + \Omega_2$. In fact this result does not depend on the number of bosons involved because from (2.2.6), we get

$$F(\alpha_1,\alpha_2,n_s) = Tr(H_P(B))/n_s = -(\alpha_1\sqrt{\Omega_1 - \alpha_1^2(n_s-1)} + \alpha_2\sqrt{\Omega_2 - \alpha_2^2(n_s-1)}\;)^2 \tag{2.2.12}$$

where n_s is the number of s-bosons. When $\alpha_1 = \sqrt{\frac{\Omega_1}{\Omega}}$ and $\alpha_2 = \sqrt{\frac{\Omega_2}{\Omega}}$, this formula gives the exact solution to the original fermion problem. Up to now no equations with a(11) and a(22)-bosons are relevant because we are treating just a pairing interaction, but they will be useful later on.

Next we consider a quadrupole interaction. The fermion quadrupole interaction is given by

$$B^{2\,F}_{\mu}(ab) = \frac{1}{\sqrt{5}} \sum_{m_a,m_b} (j_a m_a j_b - m_b|2\mu)(-1)^{j_b - m_b} a^{\dagger}_{j_a m_a} a_{j_b m_b} \qquad (2.2.13),$$

where a,b = 1,2 and its image is given by

$$B^{2\,B}_{\mu}(11) = \sqrt{\frac{2}{5\Omega_1}} a^{\dagger}_{\mu}(11) a(1) \sqrt{r_1} f_1 + \sqrt{\frac{2}{5\Omega_1}} \sqrt{r_1} f_1 a^{\dagger}(1) \tilde{a}_{\mu}(11)$$

$$-\sqrt{\frac{\Omega_1}{10}} [a^{\dagger}(11) \otimes \tilde{a}(11)]^2_{\mu} (1 - \frac{N_1}{\Omega_1}) f_{2a} \qquad (2.2.14)$$

where

$$f_1 = \frac{1}{\sqrt{r_{11} - \frac{1}{\Omega_1}}}, \qquad (2.2.15)$$

$$f_{2a} = 10 \sqrt{\frac{2}{\Omega_1}} \left\{ \begin{matrix} 2 & 2 & 2 \\ j_1 & j_1 & j_1 \end{matrix} \right\} \frac{1}{r_{11}}, \qquad (2.2.16)$$

$$r_{11} = r_1 + \frac{n_1}{\Omega_1} = 1 - \frac{2n_{11}}{\Omega_1}, \qquad (2.2.17)$$

$$N_1 = 2n_1 + 2n_{11}, \qquad (2.2.18)$$

and the quadrupole interaction is defined as

$$H_{Q.Q} = -(B^2(11) + B^2(22)) \cdot (B^2(11) + B^2(22)) \qquad (2.2.19)$$

which is diagonalizable for fermions and bosons. Here we have considered only up to L=2 bosons. $B^2_{\mu}(22)$ can be easily obtained from (2.2.14) by replacing shell label 1 by 2. The quadrupole operator $B^2_{\mu}(12)$ does not appear here because the difference between j_1 and j_2 is 4.

Again we select collective bosons and compare the results of the *collective Hamiltonian* with those given by the BKL. We now calculate the collective bosons via Klein's method. As before we write

$$a^{\dagger}(1) = \alpha_1\ s^{\dagger} \quad , \quad a^{\dagger}(2) = \alpha_2\ s^{\dagger}$$

$$a^{\dagger}(11) = \beta_1\ d^{\dagger} \quad , \quad a^{\dagger}(22) = \beta_2\ d^{\dagger} \qquad (2.2.20)$$

with the normalization conditions

$$\alpha_1^2 + \alpha_2^2 = 1 \quad , \quad \beta_1^2 + \beta_2^2 = 1 \qquad (2.2.21)$$

By minimizing $F(\alpha_1, \alpha_2, \beta_1, \beta_2, 1) = Tr(H_{Q.Q})$ for n=1 boson in terms of α_1 and α_2 under the constraints (2.2.21), we obtain

$$\alpha_1 = \beta_1 = 1 \qquad (2.2.22)$$

$$\alpha_2 = \beta_2 = 0 \qquad (2.2.23)$$

and this result turns out to be correct also for n=2 bosons. The quadrupole Hamiltonian (x = 0) can now be written in terms of collective bosons only and then diagonalized. It is important to notice that expressions (2.2.6) and (2.2.19) are expressed in terms of collective bosons and in this case collective bosons are $a(1)$ and $a(11)$. Again normal ordering is important when rewriting the collective Hamiltonian and all square root type factors which depend on boson numbers must be placed in the middle of each term of the Hamiltonian, i.e., between creation and annihilation operators.

The exact pairing plus quadrupole and the respective collective KV Hamiltonians are diagonalized for x=1.0 (pure pairing), x=0.7 and x=0.3 , x=0.1, and x=0.0 (pure quadrupole) for n=1 boson (vide table IV) and for n=2 bosons (vide table V). Following the procedure we have used to select collective bosons for the pairing and for the quadrupole interaction, one can see that the values of α's and β's depend on the interaction involved in the calculation and they are different for different values of x, as can be seen in table VI. The results are extremely good especially for the ground states. In the BKL, when n=1 boson, the results for L=0 states are exact, but they are twice smaller when compared to exact ones for L=2. As we discussed in ref. 2, this is due to neglecting L=4 bosons in the BKL method.

Although the results are good for n=1 and n=2 boson systems, KV method has a drawback that calculating traces becomes very difficult as the number of bosons increases, especially the number of d-bosons. This costs almost the same labour as diagonalizing the original BKL hamiltonian. Another method which we now consider is a number conserved Tamm Dancoff method (NTD). In this method we determine the collective S pair by satisfying

$$\delta < S^n|H|S^n >= 0 \tag{2.2.24}$$

where

$$S^\dagger = \alpha_1\, a^\dagger(1) + \alpha_2\, a^\dagger(2) \tag{2.2.25}$$

with normalization condition: $\alpha_1^2 + \alpha_2^2 = 1$. Using the same S pair, we determine the D pair by requiring

$$\delta < S^{n-1}D|H|S^{n-1}D >= 0 \tag{2.2.26}$$

where

$$D^\dagger = \beta_1\, a^\dagger(11) + \beta_2\, a^\dagger(22) \tag{2.2.27}$$

with $\beta_1^2 + \beta_2^2 = 1$ This method has an advantage that these two formulae (eqs.(2.2.24) and (2.2.26)) are easy to evaluate even for a general n-boson system. A disadvantage of this method is that it is assumed to be valid only in spherical and vibrational regions. Since the non-collective L=0 boson can be written as

$$T^\dagger = \alpha_2\, a^\dagger(1) - \alpha_1\, a^\dagger(2) \ , \tag{2.2.28}$$

we get

$$a^\dagger(1) = \alpha_1 S^\dagger + \alpha_2 T^\dagger \tag{2.2.29}$$

$$a^\dagger(2) = \alpha_2 S^\dagger - \alpha_1 T^\dagger \tag{2.2.30}$$

After obtaining α's and β's, we rewrite the original hamiltonian in terms of collective bosons and non-collective ones. Keeping the terms of collective bosons , we get a collective hamiltonian in the same way as the KV procedure. The results are shown in Tables IV and V under the column NTD . In table VI α's and β's determined respectively by equations (2.2.24) and (2.2.26) are listed.

3.Conclusion

Two simple mapping applications have been performed in this work. Firstly we have applied the trace invariant method for selecting collective bosons to a Dyson mapping in the sd shell. The results obtained after the diagonalization of a quadrupole Hamiltonian written in terms of collective bosons have exactly reproduced the low-lying states of the full Dyson Hamiltonian.

Secondly we have applied the trace invariant method and the number conserved Tamm Dancoff method to the BKL mapping for two j-shells with $|j_1 - j_2| = 4$. A pairing plus quadrupole Hamiltonian has been diagonalized. For the pairing case (n=1 boson and n=2 bosons), the BKL has exactly reproduced the fermion results as it should be expected. Both collective Hamiltonian results are approximately the same.

It is worth mentioning that the selection of collective bosons depends on the interaction considered, as can be seen by equations (2.2.11) and (2.2.23), (2.2.24) where collective bosons are written respectively for the pairing and the quadrupole interaction.

Comparing the results obtained by the *collective Hamiltonians* with the BKL results either for n=1 or n=2 bosons, we can see that the selection of collective bosons gives us reasonable results independently of the method used.

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References

1) F. Iachello and A. Arima, *"The Interacting Boson Model"* (Cambridge University Press, Cambridge, 1987)

D. Bonatsos, *"Interacting Boson Models of Nuclear Structure"* (Clarendon, Oxford, 1988)

2) D. P. Menezes, N. Yoshinaga and D. Bonatsos, *Phys. Lett.* **221B** (1989) 103

N. Yoshinaga, *Nucl. Phys.* **A503** (1989) 65

N. Yoshinaga, *Nucl. Phys.* **A493** (1989) 323

N. Yoshinaga, A. Arima and T. Otsuka, *Phys. Lett.* **143B** (1984) 5

3) A. Klein and M. Vallières, *Phys. Lett.* **98B** (1981) 5

4) D. Bonatsos and A. Klein, *Ann. Phys.* **169** (1986) 61

5) J.P. Elliott, *Proc. Roy. Soc.* **A245** (1958) 128

J.P. Elliott, *Proc. Roy. Soc.* **A245** (1958) 562

M. Harvey, *Adv. Nucl. Phys.* (M. Baranger and E. Vogt, Eds.) **vol. 1** p.67 , Plenum, New York, 1968

6) D. Bonatsos, A. Klein and C.T. Li, *Nucl. Phys.* **A425** (1984)521

7) D. P. Menezes, D. Bonatsos and A. Klein, *Nucl. Phys.* **A474** (1987) 381

D. Bonatsos, D. P. Menezes and A. Klein, *J. Phys.* **G14** (1988) L45

Table I

L	$H_{Elliott}$	H_{Dyson}
0	-14.0	-14.0
	-5.0	-5.0
2	-11.75	-11.75
	-2.75	-2.75
4	-6.5	-6.5

Table I - Quadrupole hamiltonian diagonalization for n=1 boson (2 fermions)

Table II

Table II-a : SU(3) rep. of four-fermion states in the sd shell

U(6)	(λ, μ)
[4]	(80) , (42) , (04) ,(20)
[31]	(61) , (42) , (23) , (31) , (12) , (20)
[22]	(42) , (31) , (04) , (20)
[211]	(50) , (23) , (31) , (12) , (01)
[1111]	(12)

Table II-b : SU(3) rep. of two-boson states in the sd shell. Degeneracies are indicated after SU(3) representations.

(λ, μ)
(80) ,(42) 2 , (04) 2 , (31) , (20) 2

Table III

L	$H_{Elliott}$	H_{Dyson}	KV
0	-44.00*	-44.00	-44.00
	-23.0	-23.0	-23.0
	-14.0	-14.0	-14.0
	-5.00	-5.00	
2	-41.75*	-41.75	-41.75
	-20.75	-20.75	-20.75
	-11.75	-11.75	-11.75
	-10.25	-10.25	
	-2.75	-2.75	
1	-11.75	-11.75	—
3	-18.50	-18.50	-18.50
	-8.00	-8.00	

Table III - Quadrupole interaction diagonalization for n=2 bosons (4 particles). Some states with asterisks (*) correspond to spurious states because they do not appear in the fermion space if we do not consider the isospin space.

Table IV

x	L	Exact	BKL	KV	NTD
1.0	0	-22.00	-22.00	-22.00	-22.00
		0.0	0.0		
0.7	0	-15.45	-15.45	-15.45	-15.45
	2	-0.06	-0.03	-0.03	- 0.03
0.3	0	-6.73	-6.73	-6.73	-6.73
	2	-0.15	-0.07	-0.07	-0.07
0.1	0	-2.36	-2.36	-2.36	-2.36
	2	-0.19	-0.09	-0.09	-0.09
0.0	0	-0.22	-0.22	-0.22	-0.22
		-0.15	-0.15		
	2	-0.21	-0.10	-0.10	-0.10
		-0.15	-0.07		

Table IV - Pairing plus quadrupole Hamiltonian diagonalization for n=1 boson (2 particles) and different values of x

Table V

x	L	Exact	BKL	KV	NTD
1.0	0	-42.0	-42.00	-42.00	-42.00
		-20.0	-20.00	0.00	0.00
		0.00	0.00		
	2	-20.0	-20.00	-20.0	-19.97
		-20.0	-20.00	0.00	
		0.00	0.00		
0.7	0	-29.50	-29.50	-29.50	-29.50
		-14.10	-14.10	-0.10	-0.07
		- 0.19	-0.15		
	2	-14.12	-14.10	-14.10	-14.08
		-14.10	-14.07	-0.06	-0.04
		- 0.18	-0.12		
0.3	0	-12.84	-12.84	-12.84	-12.84
		- 6.24	- 6.24	-0.23	-0.19
		- 0.44	-0.35		
	2	- 6.29	- 6.23	- 6.23	-6.23
		- 6.23	- 6.17	-0.12	-0.11
		- 0.42	-0.28		
0.1	0	- 4.52	-4.52	- 4.51	- 4.51
		- 2.31	- 2.31	-0.29	-0.25
		- 0.55	-0.45		
	2	- 2.38	- 2.30	- 2.30	-2.30
		- 2.29	- 2.23	-0.15	-0.14
		- 0.54	-0.36		
0.0	0	- 0.65	- 0.54	- 0.54	-0.54
		- 0.53	- 0.50	-0.17	-0.17
		- 0.48	-0.39		
	2	- 0.64	- 0.43	-0.43	-0.43
		- 0.55	- 0.39	-0.12	-0.12
		- 0.52	-0.32		

Table V - Pairing plus quadrupole Hamiltonian diagonalization for n=2 bosons (4 particles) and different values of x

Table VI

n	x	$\alpha_1(KV)$	$\beta_1(KV)$	$\alpha_1(NTD)$	$\beta_1(NTD)$
1	1.0	0.64		0.64	
	0.7	0.64	0.81	0.64	1.00
	0.3	0.64	0.81	0.64	1.00
	0.1	0.65	0.84	0.65	1.00
	0.0	1.00	1.00	1.00	1.00
2	1.0	0.64		0.64	
	0.7	0.64	1.00	0.64	0.39
	0.3	0.64	1.00	0.64	0.72
	0.1	0.64	1.00	0.65	0.78
	0.0	1.00	1.00	1.00	1.00

Table VI - Minimization parameters (α's and β's) for different values of x. In KV they were calculated for L=0 and in NTD α's were calculated for L=0 and β's for L=2. α_2 and β_2 are calculated by normalization conditions.