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CONSERVATION LAWS FOR FERMIONS IN SPACES WITH TORSION

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ABSTRACT

In General Relativity there is a conservation law, expressed in terms of the (symmetric) energy–momentum tensor, associated to each isometry of the space–time. Using Noether's theorem we prove the validity of these conservation laws in the more complicated case of Einstein–Cartan theory with spinorial matter. The proof is much harder because the energy–momentum tensor is not symmetric.

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1. Introduction

The theorem of Emmy Noether(1) connecting symmetries and conservation laws is a result of great importance and beauty. The original memoir, of which an English translation is now available(2), is still unsurpassed for generality and depth. On the other hand, the development of new theories opened the way to new applications of Noether's results which can be very useful when the formalism is heavy. This is the case, for instance, of space–times with torsion(3), like those of the Einstein–Cartan theory of gravity(4), particularly for fermionic matter. In this paper we address this problem and, using a lucid strategy due to Jackiw(5), succeed in obtaining simple, yet general, results.

The main applications of Noether's theorem are in classical physics, of course. The study of the generalized symmetries of the Korteweg–de Vries equation is a recent conquer(6), at least as far as this method is concerned. That was our first motivation to undertake this research, and the stimulus came from Prof. J.J. Giambiagi, who has since long been championing the study of fluid mechanics as a branch of modern theoretical physics. It is a pleasure to dedicate to him this paper.

2. The Flat Case

The classical action which describes our system (for the moment restricted to flat space–time) is written

$$S = \int d^4x\, \mathcal{L}(\phi,\partial_\mu\phi) \quad , \tag{2.1}$$

$\mathcal{L}$ being the Lagrangian density, a function of some fields ϕ and of their derivatives $\partial_\mu\phi$. To start with, ϕ will be a scalar, in order to reveal most clearly the structure of the theorem.

An infinitesimal transformation of the fields

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \delta\phi(x) \tag{2.2}$$

induces an infinitesimal variation $\delta\mathcal{L}$ in the Lagrangian. The transformation is a symmetry when it can be shown, *without using the equations of motion*, that

$$\delta\mathcal{L}(x) = \partial_\mu \Lambda^\mu \tag{2.3}$$

where Λ^μ is some 4–vector. That is to say, $\delta\mathcal{L}$ has the form given in Eq. (2.3) for all field configurations, not just for those which are solutions of the equations of motion. Example: translations. Consider the infinitesimal transformations

$$x'^\mu = x^\mu + \varepsilon^\mu \quad ,$$

ε^μ being an infinitesimal constant 4–vectors (infinitesimal translations). They induce on $\phi(x)$ a transformation $\delta\phi(x)$ to be computed now. Taking $\phi(x)$ to be a scalar under translations, one has

$$\phi'(x') = \phi(x) \quad . \tag{2.4}$$

Power expansion on ε^μ gives

$$\phi'(x') = \phi'(x) + (x'-x)^\lambda \, \partial_\lambda \, \phi(x)$$

or

$$\phi'(x') = \phi'(x) + \varepsilon^\lambda \, \partial_\lambda \, \phi(x) \tag{2.5}$$

which, combined with (2.4), gives

$$\delta\phi(x) = \phi'(x) - \phi(x) = -\varepsilon^\lambda \, \partial_\lambda \, \phi \quad . \tag{2.6}$$

Suppose

$$\mathcal{L} = (1/2) \, \partial^\mu \phi \, \partial_\mu \phi - (m^2/2)\phi^2 \quad , \tag{2.7}$$

so that,

$$\delta\mathcal{L}(x) = \partial^\mu \phi \, \partial_\mu \, \delta\phi - m^2 \phi \delta\phi \quad ; \tag{2.8}$$

using (2.6),

$$\delta\mathcal{L} = -\varepsilon^\lambda \, \partial_\lambda \left\{(1/2) \, \partial^\mu \phi \, \partial_\mu \phi - (m^2/2)\phi^2\right\} = -\varepsilon^\lambda \, \partial_\lambda \, \mathcal{L} \quad . \tag{2.9}$$

Finally, as ε^λ is a constant,

$$\delta\mathcal{L} = \partial_\lambda(-\varepsilon^\lambda \, \mathcal{L}) \quad , \tag{2.10}$$

which shows that translations are symmetries of the system described by the Lagrangian (2.7). We will, henceforth, call quantities like $\delta\phi(x)$ the "form variation" of the field in question.

The Noether theorem asserts that to each continuous symmetry there corresponds a current which satisfies a continuity equation. Furthermore, it gives an explicit expression for that current. Suppose $\delta\phi$ is the symmetry transformation. Then there is Λ^μ such that

$$\delta\mathcal{L}(x) = \partial_\mu \Lambda^\mu \quad . \tag{2.3}$$

An independent computation of $\delta\mathcal{L}$, *now using the equations of motion*, will now be done:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \, \partial_\mu \, \delta\phi \tag{2.11}$$

and the equations of motion are

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \quad . \tag{2.12}$$

Using (2.12) into (2.11) gives

$$\delta\mathcal{L} = \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \, \delta\phi \right] \quad . \tag{2.13}$$

Subtracting (2.13) from (2.3) one gets

$$\partial_\mu \left\{ \Lambda^\mu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \, \delta\phi \right\} = 0 \quad . \tag{2.14}$$

This is the Noether theorem. The 4–vector

$$j^\mu = \Lambda^\mu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \tag{2.15}$$

is the Noether current associated to the symmetry $\delta\phi$.

It is a simple matter to see that the conservation law associated to translations is

$$\partial_\mu T^\mu{}_\nu = 0 \tag{2.16}$$

where

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta^\mu{}_\nu\,\mathcal{L} \tag{2.17}$$

is the canonical energy–momentum tensor.

3. The Curved Case

An infinitesimal transformation in curved space–time

$$x'^\mu = x^\mu + \xi^\mu(x) \tag{3.1}$$

induces on a scalar field $\phi(x)$ the same form variation we have met before,

$$\delta\phi(x) = -\,\xi^\lambda\,\partial_\lambda\,\phi \ . \tag{3.2}$$

Let us compute the form variation induced on the metric tensor $g^{\mu\nu}(x)$. From

$$g'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\,\frac{\partial x'^\nu}{\partial x^\beta}\,g^{\alpha\beta}(x) \tag{3.3}$$

it follows, for the infinitesimal transformations (3.1), that

$$g'^{\mu\nu}(x') = g^{\mu\nu}(x) + \partial^\mu\,\xi^\nu + \partial^\nu\,\xi^\mu \tag{3.4}$$

and, by Taylor expansion around x^μ,

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x) + \xi^\lambda\,\partial_\lambda\,g^{\mu\nu}(x) \ . \tag{3.5}$$

Using both, one arrives at

$$\delta g^{\mu\nu}(x) = -\,\xi^\lambda\,\partial_\lambda\,g^{\mu\nu} + \partial^\mu\,\xi^\nu + \partial^\nu\,\xi^\mu \tag{3.6}$$

or, equivalently,

$$\delta g^{\mu\nu}(x) = \xi^{\mu;\nu} + \xi^{\nu;\mu} \ . \tag{3.7}$$

Vector fields $\xi^\mu(x)$ which generate transformations for which $\delta g^{\mu\nu} = 0$ are called Killing fields. They are, therefore, characterized by

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \ . \tag{3.8}$$

A transformation of type (3.1) with a ξ^μ which is Killing, is called an isometry of the space–time. Observers connected by such a transformation observe identical metric relationships in space–time, and hence the same gravitational field.

We have still to determine the form variation of a Dirac spinor. This is a harder task. The inspiration will come from the Dirac equation in curved space–times.

4. Dirac Equation in Einstein–Cartan Spaces

The Dirac action in spaces with torsion (see Refs. (3), (4), (7)) reads

$$S = \int d^4x\,\sqrt{-g}\,\mathcal{L}$$

$$\mathcal{L} = \frac{i}{2}\,e_a{}^\mu\,(\bar\psi\,\gamma^a\,\nabla_\mu - \nabla_\mu\,\bar\psi\,\gamma^a\,\psi) - m\bar\psi\psi \tag{4.1}$$

where $e_a^{\ \mu}$ are local tetrads, γ^a are the usual Dirac matrices and ∇_μ is the covariant derivative for fermions, which reads

$$\nabla_\mu \psi = \left[\partial_\mu + \tfrac{1}{4}\,\omega_{\mu ab}\,\gamma^{[a}\,\gamma^{b]}\right]\psi \tag{4.2}$$

and

$$\nabla_\mu \bar\psi = \partial_\mu \bar\psi - \tfrac{1}{4}\,\omega_{\mu ab}\,\bar\psi\,\gamma^{[a}\,\gamma^{b]} \ . \tag{4.3}$$

Recall that

$$\Omega_{ab}^{\ \ c} = e_a^{\ \mu}\, e_b^{\ \mu}\, \partial_{[\mu}\, e_{\nu]}^{\ c} \tag{4.4}$$

and that

$$\omega_{abc} = \Omega_{abc} - \Omega_{bca} + \Omega_{cab} - S_{abc} - S_{bca} + S_{cab} \tag{4.5}$$

and

$$\omega_{\mu ab} = e_\mu^{\ c}\,\omega_{cab} \ , \tag{4.6}$$

S_{abc} being tetrad components of the torsion tensor

$$S_{\mu\nu}^{\ \ \lambda} = \Gamma_{[\mu\nu]}^{\ \ \ \lambda} \ .$$

See the Appendix for a collection of formulas and conventions.

An equivalent way of writing (4.1) is

$$\mathcal{L} = (i/2)(\bar\psi\, e_a^{\ \mu}\,\gamma^a\,\partial_\mu\,\psi - \partial_\mu\,\bar\psi\, e_a^{\ \mu}\,\gamma^a\,\psi) + (i/4)\,\omega_{abc}\,\bar\psi\,\gamma^{[a}\gamma^b\gamma^{c]}\,\psi - m\bar\psi\psi \tag{4.7}$$

5. Field Equations

We now let the Dirac field to interact with the (Einstein–Cartan) gravitational field by writing the action

$$S = \int d^4x\,\sqrt{-g}\,(\mathcal{L}_m + \mathcal{L}_g) \tag{5.1}$$

where $\mathcal{L}_m$ will be the Dirac Lagrangian and[4]

$$\mathcal{L}_g = (-1/2\pi k)R \tag{5.2}$$

which, though formally identical to Einstein's action, differs from it in that torsion is present. Eq. (5.1) therefore reads

$$S = \int d^4x\, e\left\{(-1/2k)R + (i/2)\,e_a^{\ \mu}(\bar\psi\,\gamma^a\,\nabla_\mu\,\psi - \nabla_\mu\,\bar\psi\,\gamma^a\,\psi) - m\bar\psi\psi\right\} . \tag{5.3}$$

Field equations follow in the usual way, by requiring that the variations with respect to $g_{\mu\nu}$, $K_{\mu\nu}^{\ \ \lambda}$ (the contorsion tensor) and ψ vanish. Putting

$$\frac{\delta(e\mathcal{L}_m)}{\delta g^{\mu\nu}} = \frac{2}{e}\,t_{\mu\nu} \tag{5.4}$$

and

$$(\delta(e\mathcal{L}_m)/\delta K_{\mu\nu}^{\ \ \lambda}) = e\,\tau_\lambda^{\ \mu\nu} \tag{5.5}$$

one gets (see the Appendix)

$$G_{\mu\nu} - \overset{*}{\nabla}_\lambda(T_{\mu\nu}^{\ \ \lambda} - T_{\nu\ \mu}^{\ \lambda} + T^\lambda_{\ \mu\nu}) = k\,t_{\mu\nu} \tag{5.6}$$

and

$$T^{\mu\nu\lambda} = -k\,\tau^{\mu\nu\lambda} \ . \tag{5.7}$$

Defining a new energy–momentum tensor

$$\Sigma_{\mu\nu} = t_{\mu\nu} - \overset{*}{\nabla}_\lambda(\tau_{\mu\nu}^{\ \ \lambda} - \tau_{\nu\ \mu}^{\ \lambda} + \tau^\lambda_{\ \mu\nu}) \tag{5.8}$$

one is able to rewrite (5.6) as

$$G_{\mu\nu} = k\,\Sigma_{\mu\nu} \tag{5.9}$$

which looks like Einstein's equation. The tensor $\Sigma_{\mu\nu}$, however, is not symmetric, the same as $G_{\mu\nu}$.

The Dirac equation, obtained by requiring that

$$\delta(e\mathcal{L}_m)/\delta\bar{\psi} = 0 \quad , \tag{5.10}$$

reads

$$i\,\gamma^a e_a{}^{\mu} \nabla_\mu \psi + \tfrac{1}{2} K_{\lambda\mu}{}^{\lambda} \psi - m\psi = 0 \quad . \tag{5.11}$$

6. The Noether Theorem for Spinors

We now apply the strategy of Section 2 to prove Noether's theorem for spinor fields in Einstein–Cartan theory. We look for space–time symmetries which lead to *tensorial* conservation laws. In General Relativity these symmetries turn out to be the isometries of the space–time(8), when they exist. We will see that the same is true in our more general setting, provided we slightly modify the definition of isometry, in the presence of torsion.

The basic quantities are the form variations of all fields present in the action. For an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{6.1}$$

one has

$$\delta\phi(x) = -\xi^{\lambda}\partial_\lambda \phi \tag{6.2}$$

$$\delta\psi(x) = -\xi^{\lambda}\nabla_\lambda \psi \tag{6.3}$$

$$\delta(\nabla_\mu \psi(x)) = -\nabla_\lambda \psi \nabla_\mu \xi^{\lambda} - \xi^{\lambda}\nabla_\lambda\nabla_\mu \psi - 2\,\xi^{\lambda} S_{\lambda\mu}{}^{\nu}\nabla_\nu \psi \tag{6.4}$$

$$\delta e_a{}^{\mu} = \nabla_\lambda \xi^{\mu} e_a{}^{\lambda} + 2\,S_{\lambda\nu}{}^{\mu}\xi^{\lambda} e_a{}^{\nu} = \partial_\lambda \xi^{\mu} e_a{}^{\lambda} + \Gamma_{\lambda\nu}{}^{\mu}\xi^{\lambda} e_a{}^{\nu} \tag{6.5}$$

for, respectively, scalar fields, spinors and their derivatives, and tetrads.

The next step is to show that isometries are symmetries. Consider first the Dirac action

$$S_m = \int d^4x\, e(x)\, \mathcal{L}(\psi, \bar{\psi}, \nabla_\mu\psi, \nabla_\mu\bar{\psi}, e_a{}^{\mu}) \quad , \tag{6.6}$$

whose form variation we start to compute.

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\partial\psi}\delta\psi + \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\bar{\psi}} + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\psi}\delta\nabla_\mu\psi + \delta\nabla_\mu\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\bar{\psi})} + \frac{\partial\mathcal{L}}{\partial e_a{}^{\mu}}\delta e_a{}^{\mu}$$

$$\begin{aligned}\delta\mathcal{L} = & -\xi^{\lambda}\left\{\frac{\delta\mathcal{L}}{\partial\psi}\nabla_\lambda\psi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\psi)}\nabla_\lambda\nabla_\mu\psi + 2\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\psi)}S_{\lambda\mu}{}^{\nu}\nabla_\nu\psi\right\} - \\ & -\xi^{\lambda}\left\{\nabla_\lambda\bar{\psi}\frac{\delta\mathcal{L}}{\partial\bar{\psi}} + \nabla_\lambda\nabla_\mu\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\bar{\psi})} + 2\nabla_\nu\bar{\psi}\frac{\partial\mathcal{L}}{\partial\nabla_\mu\bar{\psi}}S_{\lambda\mu}{}^{\nu}\right\} - \\ & -\nabla_\mu\xi^{\lambda}e_a{}^{\mu}\left[\tfrac{i}{2}\bar{\psi}\gamma^a\nabla_\lambda\psi - \tfrac{i}{2}\nabla_\lambda\bar{\psi}\gamma^a\psi\right] + \frac{\partial\mathcal{L}}{\partial e_a{}^{\mu}}(\nabla_\lambda\xi^{\mu}e_a{}^{\lambda} + S_{\lambda\nu}{}^{\mu}\xi^{\lambda}e_a{}^{\nu}) \quad .\end{aligned} \tag{6.7}$$

Notice that

$$\tfrac{i}{2}(\bar{\psi}\gamma^a\nabla_\lambda\psi - \nabla_\lambda\bar{\psi}\gamma^a\psi) = \frac{\partial\mathcal{L}}{\partial\, e_a{}^{\lambda}}$$

so that (6.7) may be rewritten as

$$\delta\mathcal{L} = -\xi^{\lambda}\left[\frac{\delta\mathcal{L}}{\partial\psi}\nabla_\lambda\psi + \frac{\partial\mathcal{L}}{\partial\,\nabla_\mu\,\psi}\nabla_\lambda\nabla_\mu\psi + \nabla_\lambda\bar{\psi}\frac{\delta\mathcal{L}}{\partial\bar{\psi}} + \nabla_\lambda\nabla_\mu\frac{\partial\mathcal{L}}{\partial\,\nabla_\mu\,\bar{\psi}}\right] -$$

$$-2\xi^\lambda \left[e_a{}^\mu \frac{\partial \mathcal{L}}{\partial\, e_a{}^\nu} S_{\lambda\mu}{}^\nu - e_a{}^\nu \frac{\partial \mathcal{L}}{\partial\, e_a{}^\mu} S_{\lambda\nu}{}^\mu \right] \quad . \tag{6.8}$$

The last term vanishes, and we have

$$\delta\mathcal{L} = -\xi^\lambda \left[\frac{\delta\mathcal{L}}{\partial\psi} \nabla_\lambda \psi + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\psi} \nabla_\lambda \nabla_\mu \psi + \nabla_\lambda \bar\psi \frac{\delta\mathcal{L}}{\partial\bar\psi} + \nabla_\lambda \nabla_\mu \frac{\partial\mathcal{L}}{\partial\nabla_\mu\bar\psi} + \frac{\partial\mathcal{L}}{\partial\, e_a{}^\mu} \nabla_\lambda e_a{}^\mu \right] \tag{6.9}$$

where the last term can be added because $\nabla_\lambda e_a{}^\mu = 0$. But then (6.9) reads

$$\delta\mathcal{L} = -\xi^\lambda \nabla_\lambda \mathcal{L} = -\xi^\lambda \partial_\lambda \mathcal{L} \quad . \tag{6.10}$$

Suppose the transformation (6.1) is an isometry. For us this is defined by the conditions

$$\delta g^{\mu\nu} = 0 \tag{6.11}$$

and

$$\delta K_{\mu\nu}{}^\lambda = 0 \quad , \tag{6.12}$$

so that $\delta e = 0$ and $\partial_\lambda(e\xi^\lambda) = 0$. Equation (6.10) then leads to

$$\delta(e\mathcal{L}) = e\delta\mathcal{L} = -e\,\xi^\lambda \partial_\lambda \mathcal{L} = -\partial_\lambda(e\,\xi^\lambda \mathcal{L}) \tag{6.13}$$

showing that isometries aew symmetries of the Dirac action.

Now, for the whole action, S_g+S_m , one has

$$\delta S = \int d^4x \left[\delta(e\mathcal{L}) - \frac{1}{2k}\,\delta(eR) \right]$$

$$= \int d^4x\, e \Big[\Big\{ -\frac{1}{2k} \left[G_{\mu\nu} - \overset{*}{\nabla}_\lambda (T_{\mu\nu}{}^\lambda - T_\nu{}^\lambda{}_\mu + T^\lambda{}_{\mu\nu}) \right] + \frac{1}{2} t_{\mu\nu} \Big\} \delta g^{\mu\nu} +$$

$$+ \left[\frac{1}{k} T_\lambda{}^{\nu\mu} + \tau_\lambda{}^{\nu\mu} \right] \delta K_{\mu\nu}{}^\lambda - e\,\xi^\lambda \partial_\lambda \mathcal{L} \Big] \quad , \tag{6.14}$$

so that, for isometries,

$$\delta S = -\int d^4x\, \partial_\lambda(e\,\xi^\lambda \mathcal{L}) \tag{6.15}$$

showing that isometries are symmetries of the total action.

To proceed, we now compute δS in a different way, making use of the equations of motion.

$$\delta S_m = \int d^4x\, e \left[\frac{\partial\mathcal{L}}{\partial\psi} \delta\psi + \delta\bar\psi \frac{\partial\mathcal{L}}{\partial\bar\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \delta\, \partial_\mu \psi + \delta\, \partial_\mu \bar\psi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} \right] +$$

$$+ \int d^4x \left[\frac{\partial(e\mathcal{L})}{\partial K_{\mu\nu}{}^\lambda} \delta K_{\mu\nu}{}^\lambda + \frac{\partial(e\mathcal{L})}{\partial\, e_a{}^\mu} \delta e_a{}^\mu \right] \tag{6.16}$$

and this last integral is

$$\int d^4x\, e(\tau_\lambda{}^{\nu\mu} \delta K_{\mu\nu}{}^\lambda + \frac{1}{2} t_{\mu\nu}\, \delta g^{\mu\nu}) \quad .$$

For $\delta S = \delta S_g + \delta S_m$ we then have

$$\delta S = \int d^4x \sqrt{-g} \Big\{ \left[\tau_\lambda{}^{\nu\mu} + \frac{1}{k} T_\lambda{}^{\nu\mu} \right] \delta K_{\mu\nu}{}^\lambda +$$

$$+ \delta g^{\mu\nu} \Big[-\frac{1}{2k} \left[G_{\mu\nu} - \overset{*}{\nabla}_\lambda (T_{\mu\nu}{}^\lambda - T_\nu{}^\lambda{}_\mu + T^\lambda{}_{\mu\nu}) \right] + \frac{1}{2} t_{\mu\nu} \Big] +$$

$$+ \frac{\partial \mathcal{L}}{\partial \psi}\, \delta\psi + \delta\bar{\psi}\, \frac{\partial \mathcal{L}}{\partial \bar{\psi}} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, \delta\, \partial_\mu \psi + \delta\, \partial_\mu \bar{\psi}\, \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \Bigg\} \ . \tag{6.17}$$

Using now all the equations of motion δS is reduced to

$$\delta S = \int d^4x\, \partial_\mu \left[e \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi}\, \delta\psi + e\delta\bar{\psi}\, \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \right]$$

$$= - \int d^4x\, \partial_\mu \left\{ e\, \xi^\lambda \left[\frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi} \nabla_\lambda \psi + \nabla_\lambda \bar{\psi} \frac{\partial \mathcal{L}}{\partial \nabla_\mu \bar{\psi}} \right] \right\} \ . \tag{6.18}$$

Subtracting (6.18) from (6.15) one has

$$\int d^4x\, \partial_\mu \left\{ e \xi^\lambda \left[\frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi} \nabla_\lambda \psi + \nabla_\lambda \bar{\psi} \frac{\partial \mathcal{L}}{\partial \nabla_\mu \bar{\psi}} - \delta^\mu{}_\lambda\, \mathcal{L} \right] \right\} = 0 \ , \tag{6.19}$$

showing that

$$\partial_\mu \left\{ e\, \xi^\lambda\, \theta_\lambda{}^\mu \right\} = 0 \tag{6.20}$$

where

$$\theta_\lambda{}^\mu = \frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi} \nabla_\lambda \psi + \nabla_\lambda \bar{\psi} \frac{\partial \mathcal{L}}{\partial \nabla_\mu \bar{\psi}} - \delta^\mu{}_\lambda\, \mathcal{L}$$

$$= \frac{i}{2} e_a{}^\mu \left(\bar{\psi}\, \gamma^a\, \nabla_\lambda\, \psi - \nabla_\lambda\, \bar{\psi}\, \gamma^a\, \psi \right) - \delta^\mu{}_\lambda\, \mathcal{L} \ . \tag{6.21}$$

Equation (6.20) is Noether's theorem: every isometry

$$x'^\mu = x^\mu + \xi^\mu(x)$$

is associated to a conservation law given by (6.20). Notice that $\theta_{\lambda\mu}$ is the same as the $\Sigma_{\lambda\mu}$ of Eq. (5.8), so that it is not symmetric. In the case of General Relativity $\theta_{\lambda\mu}$ is the usual symmetric energy–momentum tensor, and Eq. (6.20) is an immediate consequence of the symmetry of $t_{\mu\nu}$ and the antisymmetry of $\xi_{\mu;\nu}$. In the case of Einstein–Cartan, however, this result can only be obtained in the garb of the theorem of Emmy Noether.

Appendix

Riemann–Cartan spaces have a (non symmetric) connection

$$\Gamma_{\mu\nu}^{\ \ \lambda} = \left\{ {\lambda \atop \mu \ \nu} \right\} - K_{\mu\nu}^{\ \ \lambda} \qquad (A.1)$$

where $K_{\mu\nu\lambda} = K_{\mu[\nu\lambda]}$ is the contorsion tensor. The torsion is given by

$$S_{\mu\nu}^{\ \ \lambda} = \Gamma_{[\mu\nu]}^{\ \ \ \lambda} \ , \qquad (A.2)$$

and one has

$$K_{\mu\nu}^{\ \ \lambda} = -S_{\mu\nu}^{\ \ \lambda} + S_{\nu\ \mu}^{\ \lambda} - S^{\lambda}_{\ \mu\nu}$$
$$S_{\mu\nu}^{\ \ \lambda} = -K_{[\mu\nu]}^{\ \ \ \lambda} \ . \qquad (A.3)$$

The modified torsion tensor is given by

$$T_{\mu\nu}^{\ \ \lambda} = S_{\mu\nu}^{\ \ \lambda} + 2\,\delta_{[\mu}^{\lambda} S_{\nu]\alpha}^{\ \ \ \alpha} \ . \qquad (A.4)$$

We assume that

$$\nabla_\lambda g_{\mu\nu} = 0 \ , \qquad (A.5)$$

∇_λ being the covariant derivative with the connection (A.1). The curvature and Ricci tensors are

$$R_{\mu\nu\alpha}^{\ \ \ \ \beta} = 2\,\partial_{[\mu}\Gamma_{\nu]\alpha}^{\ \ \ \beta} + 2\,\Gamma_{[\mu|\rho}^{\ \ \ \ \beta}\Gamma_{|\nu]\alpha}^{\ \ \ \ \rho} \qquad (A.6)$$

and

$$R_{\mu\nu} = R_{\lambda\mu\nu}^{\ \ \ \ \lambda} \ . \qquad (A.7)$$

The identity

$$\nabla_{[\mu} R_{\nu\lambda]\alpha}^{\ \ \ \ \ \beta} = 2\,S_{[\mu\nu}^{\ \ \ \sigma} R_{\lambda]\sigma\alpha}^{\ \ \ \ \ \beta} \qquad (A.8)$$

corresponds to a Bianchi identity of riemannian geometry. The "Einstein tensor"

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \qquad (A.9)$$

is not symmetric. Instead,

$$G_{[\mu\nu]} = \overset{*}{\nabla}_\lambda T_{\mu\nu}^{\ \ \lambda} \qquad (A.10)$$

where

$$\overset{*}{\nabla}_\mu Q = \nabla_\mu Q + 2\,S_{\mu\lambda}^{\ \ \lambda} Q \ . \qquad (A.11)$$

Denoting by $R(\Gamma)$ and $R(\{\ \})$ the scalar curvatures of the connection Γ and $\{\ \}$ respectively, one has

$$R(\Gamma) = R(\{\ \}) - T_\lambda^{\ \nu\mu} K_{\mu\nu}^{\ \ \lambda} + \frac{1}{\sqrt{-g}}\,\partial_\mu \left[2\sqrt{-g}\,K_\lambda^{\ \mu\lambda}\right] \qquad (A.12)$$

From (A.12) one easily derives

$$\frac{1}{e}\,\frac{\delta(eR)}{\delta g^{\mu\nu}} = G_{\mu\nu} - \overset{*}{\nabla}_\lambda (T_{\mu\nu}^{\ \ \lambda} - T_{\nu\ \mu}^{\ \lambda} + T^{\lambda}_{\ \mu\nu}) \qquad (A.13)$$

$$\frac{1}{e}\,\frac{\delta(eR)}{\delta K_{\mu\nu}^{\ \ \lambda}} = -2\,T_\lambda^{\ \nu\mu} \qquad (A.14)$$

which are used in the text.

References

(1) E. Noether, "Invariante Variationsprobleme", Nach. zu Göttingen, 248 (1918).

(2) E. Noether, "Invariant Variation Problems", Transport Theory and Stat. Phys. 1, 186 (1971).

(3) E. Cartan, "On manifolds with an affine connection and the theory of general relativity", Bibliopolis, Naples, 1986.

(4) F.W. Hehl, P. von der Heyde, G.D. Kerlick, J.M. Nester, Revs. Mod. Phys. 48, 393 (1976).

(5) R. Jackiw, in D. Gross (ed.), "Lectures on Current Algebra and its Applications", Princeton University Press, 1972.

(6) P.J. Olver, "Applications of Lie groups to differential equations", Springer, 1986.

(7) V. de Sabbata, M. Gasperini, "Introduction to Gravitation", World Scientific, 1985.

(8) H. Fleming, Revista Brasileira de Física 17, 236 (1987).