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**FIELD THEORETICAL APPROACH TO BOSE-EINSTEIN
CONDENSATION**

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ABSTRACT

This paper presents a novel approach to Bose-Einstein condensation. We show how one can get all relevant informations on the Bose-Einstein condensation phenomena from the field theoretical Green's functions in momentum space computed at zero-momenta. One gets general expressions for the density of the zero-momentum state as a function of temperature as well as a complete prescription for determining all thermodynamical variables in the condensed phase. Explicit results are obtained by computing the generating functional of vertex functions at zero-momenta under simple approximations.

I. INTRODUCTION

The relevance of Bose-Einstein condensation for understanding the properties of liquid He^4 was pointed out first by London⁽¹⁾ who, besides proving Bose-Einstein condensation for an ideal gas, suggested that the peculiar phase transition that liquid helium undergoes at 2.19° be regarded as a Bose-Einstein condensation phenomenon. The temperature for which a finite fraction of all helium atoms will be assembled in the lowest energy state, in the ideal gas approximation, is very close the critical temperature of the λ -point. The hypothesis that He^4 , in the superfluid phase, has to do with Bose-Einstein condensation received a boost after the seminal work of Bogoliubov⁽²⁾ who showed that the "phonon"-part of spectrum of excitations of He^4 follow directly from Bose-Einstein condensation of non perfect gases. Nowadays there is a kind of consensus that superfluidity has to do with Bose-Einstein condensation.

From the above arguments it follows that the description of the condensed phase of He^4 should correspond to the description of the superfluid. The equation of state, internal energy and entropy in the condensed phase are the ones in helium superfluid. This is the main motivation for reanalysing this old, fascinating and not yet resolved (from the microscopic point of view) problem. This paper deals with an approach for tackling this problem within the context of field theory.

The approach here proposed is a variant of the Bogoliubov method. We employ the Bogoliubov method as a way of isolating the relevant degree of freedom of the condensation phenomenon (the $\vec{p} = 0$ state or the condensate). The functional integration approach allows us to write the Gibbs potential as a function of the condensate. This allows us to reduce the problem to a variational problem since the system will choose the condensate configuration which minimizes the Gibbs potential.

The next step, in order to provide a non-perturbative scheme in field theory, for determining the Gibbs potential and the whole thermodynamics, will be to relate this

thermodynamical potential to the generating functional of the vertex functions of the theory. This equivalence will be shown by employing the so-called background field method. This provides, as we shall see, a definite scheme for studying the system in the condensed phase in field theory. We will show that this framework not only provides a well defined scheme for establishing the problem of Bose-Einstein condensation but also (and this is more important) provides a well defined scheme for getting information on the system when condensation occurs. In particular, we will show how one can get an equation of state as well as expressions for the internal energy, entropy and other thermodynamical variables in the condensed phase. These quantities are expressed in terms of the Green's function, of the associated finite temperature field theory, computed at zero-momenta.

Besides establishing such a general framework we will get a more refined technique for determining the occupation of the $\beta = 0$ state as a function of the temperature. Some well known results are easily obtained from our expressions.

In order to present the framework we review some basic ideas and expressions in field theory at finite temperature in Chapter II. We review, also in this Chapter, some definitions and formulae that are useful when one is working in the Grand Canonical Ensemble. In Chapter II we formulate the Bose-Einstein condensation problem. Our formulation of the Bose-Einstein condensation leads naturally to a variational problem.

In Chapter III we show that the appropriate functional in the case of Bose-Einstein condensation is the generating functional of the vertex functions (Γ) taken at zero momenta. This leads to a definite scheme for making predictions in field theory at finite temperature. The equivalence between the Gibbs potential in the presence of the condensate and the functional Γ is established through the background field method which is also presented in Chapter III.

The formulation of the Bose-Einstein condensation in field theory is fully presented and exploited in Chapter IV. The prescription for the determination of all thermodynamical variables in the condensed phase is also exhibited in this Chapter.

Since Bogoliubov's method was formulated, and later extended by Beliaev⁽³⁾, in terms of operators we have shown the equivalence of our method and the operator approach in Chapter V. Some of our results at zero temperature (determination of $\rho_0(\mu)$) are equivalent to his.

As a simple exercise and in order to illustrate our technique, we present the description of condensation at the zero-loop level for an interacting system in Section VI.1 and the well known example of free bosons in Section VI.2. In the last case the phenomena is typically a quantum one. Some general results are also deduced in Section VI.3.

II. FIELD THEORETICAL APPROACH⁽⁴⁻⁶⁾

II.1. WIEGEL-JALICKEE REPRESENTATION AND GENERATING FUNCTIONALS

In this paper we will be concerned with a nonrelativistic system of N particles interacting among themselves through a potential $V(\mathbf{x}-\mathbf{x}')$ (we assume binary interactions) where \mathbf{x} and \mathbf{x}' are the positions of the particles. By employing the second quantization scheme, in which ψ and ψ^+ are field operators, the Hamiltonian for such a system, in the Heisenberg picture, is

$$H \equiv \int d^3\mathbf{x} \psi^+(\mathbf{x},t) \left[-\frac{\hbar^2}{2m} \nabla^2 \right] \psi(\mathbf{x},t) + \int d^3\mathbf{x} \int d^3\mathbf{x}' \psi^+(\mathbf{x},t) \psi(\mathbf{x},t) V(\mathbf{x}-\mathbf{x}') \psi^+(\mathbf{x}',t) \psi(\mathbf{x}',t) \quad (2.1)$$

where the field operator ψ and ψ^+ satisfies the usual commutation relations at equal time, that is,

$$[\psi(\mathbf{x},t), \psi^+(\mathbf{x}',t)] = \delta(\mathbf{x}-\mathbf{x}') \quad (2.2)$$

The thermodynamical properties of the system described by (2.1) should be inferred from the grand partition function Ξ

$$\Xi = \text{Tr} \left[e^{-\beta(\hat{H}-\mu\hat{N})} \right] \quad (2.3)$$

where $\beta = (T)^{-1}$, $\hat{N} = \int d^3\mathbf{x} \psi^+ \psi$ and μ is the chemical potential.

Within the functional approach the grand partition function can be written as a sum over field configurations satisfying periodic boundary conditions⁽⁴⁾

$$\psi(0, \mathbf{x}) = \psi(\beta, \mathbf{x}) \quad (2.4)$$

The functional integration representation for the Hamiltonian (2.1) has been derived by Wiegel and Jalickee⁽⁴⁾. One writes

$$\Xi = \int \mathcal{D}\psi \int \mathcal{D}\psi^* \exp -S[\psi] \quad (2.5)$$

where the action $S[\psi]$ in (2.5) is written as

$$S[\psi] = \int_0^\beta d\tau \int d^3\mathbf{x} \left[\psi^*(\tau, \mathbf{x}) \frac{\partial \psi}{\partial \tau}(\tau, \mathbf{x}) + \frac{\vec{\nabla} \psi(\mathbf{x}, \tau) \cdot \vec{\nabla} \psi(\mathbf{x}, \tau)}{2m} - \mu \psi^*(\tau, \mathbf{x}) \psi(\tau, \mathbf{x}) \right] + \int_0^\beta d\tau \int d^3\mathbf{x} \int d^3\mathbf{x}' \psi^*(\mathbf{x}, \tau) \psi(\mathbf{x}, \tau) V(\mathbf{x}-\mathbf{x}') \psi^*(\mathbf{x}', \tau) \psi(\mathbf{x}', \tau) \quad (2.6)$$

The thermodynamic properties of the system can be inferred from the Gibbs potential Ω^0 defined as

$$-\beta \Omega^0 \equiv \ln \Xi \quad (2.7)$$

That is, from Ω , defined in (2.7), one gets the equation of state from equations

$$P = - \left[\frac{\partial \Omega^0}{\partial V} \right]_{T, \mu} \quad (2.8)$$

$$N = - \left[\frac{\partial \Omega^0}{\partial \mu} \right]_{T, V} \quad (2.9)$$

whereas the internal energy (U), free energy (F) and entropy (S) are obtained from

$$U = \frac{\partial}{\partial \beta} (\beta \Omega^0) \quad (2.10)$$

$$F = \Omega^0 + N\mu \quad (2.11)$$

$$S = \beta^2 \frac{\partial \Omega^0}{\partial \beta} - \mu N \beta \quad (2.12)$$

One can study the thermodynamical properties of the system described by the Hamiltonian (2.1) in the presence of an external source J . In this case one is interested in the thermodynamical potential $\Omega(J)$ defined as

$$\Xi(J) = e^{-\Omega(J)} = \int \int \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S(\psi) - \int_0^\beta d\tau \int d^3\mathbf{x} (J^*(\mathbf{x}, \tau) \psi(\mathbf{x}, \tau) + J(\mathbf{x}, \tau) \psi^*(\mathbf{x}, \tau))} \quad (2.13)$$

The average in the ensemble of products of fields are easily obtained from $\Omega(J)$. The average of the field $\psi(x)$ (from now in x stands for (\mathbf{x}, τ)), for instance, defined by

$$\langle \psi(x) \rangle \equiv \frac{\text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} \hat{\psi}(x)}{\text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})}} \quad (2.14)$$

can be obtained from (2.13) as

$$\langle \psi(x) \rangle = \left. \frac{\partial \Omega(J)}{\partial J^*(x)} \right|_{J^*=0} \quad (2.15)$$

From (2.13) it can be seen that $\Omega(J)$ is the functional generator of the connected

Green's functions⁽²⁾. That is

$$G^{(n,m)}(x_1 \cdots x_n; y_1 \cdots y_m) = \left. \frac{\delta^{n+m} \Omega(J)}{\delta J(x_1) \cdots \delta J(x_n) \delta J^*(y_1) \delta J^*(y_m)} \right|_{J=0} \quad (2.16)$$

It is possible to introduce another thermodynamical potential $\Gamma(\psi)$ through a Legendre transformation. This Legendre transformation is⁽⁵⁾

$$\Gamma(\psi) = \Omega[J] + \int_0^\beta d\tau \int d^3\mathbf{x} (J^*(x) \psi(x) + J(x) \psi^*(x)) \quad (2.17)$$

$\Gamma(\psi)$ is the generating functional of the one-particle irreducible Green's functions (vertex functions) of the theory. One can then write

$$\Gamma(\psi) = \sum_n \sum_m \frac{1}{n!} \frac{1}{m!} \int dx_1 \cdots \int dx_n \int dy_1 \cdots \int dy_m \times \\ \times \Gamma^{(n+m)}(x_1 \cdots x_n; y_1 \cdots y_m) \psi(x_1) \cdots \psi(x_n) \psi^*(y_1) \psi^*(y_2) \cdots \psi^*(y_m) \quad (2.18)$$

In order to compute the Green's functions at finite temperatures one can use the perturbative approach. In this case the method is the same as the usual one except for some changes in the Feynman rules⁽⁷⁾.

II.2. BOSE-EINSTEIN CONDENSATION

In order to study Bose-Einstein condensation we separate out the uniform (in space) field theoretical configurations. In this way one writes, in analogy with Bogoliubov's approach⁽²⁾,

$$\psi = \psi_0 + \psi' \quad (2.19)$$

By substituting (2.19) into (2.5) one can write the Gibbs potential as a function of the Bogoliubov's condensate, that is

$$\Omega^0 = \Omega^0(T, V, \mu, \psi_0) \quad (2.20)$$

Since the number of particles with zero-momentum is given by

$$N_0 = V \rho_0 = V \psi_0^* \psi_0, \quad (2.21)$$

the dependence of Ω on ψ_0 is equivalent to the dependence of Ω on N_0 . As suggested by Glassgold, Kaufman and Watson⁽⁸⁾ the important contribution comes from the term that minimizes Ω that is

$$\left. \frac{\partial \Omega^0}{\partial N_0} \right|_{N_0} = 0 \quad (2.22)$$

or equivalently:

$$\left. \frac{\partial \Omega^0}{\partial \rho_0} \right|_{\rho_c} = 0 \quad (2.23)$$

Equations (2.22)–(2.23) or, equivalently,

$$\left. \frac{\partial \Omega^0}{\partial \psi_0} \right|_{\psi_0 = \psi_c} = 0 \quad (2.24)$$

are the basic equations for the condensation phenomena. From these equation one should be able to get

$$N_0 = N_0(\mu, T, V) \quad (2.25)$$

This equation coupled with the equation

$$N = - \frac{\partial \Omega^0}{\partial \mu} \quad (2.26)$$

allows as to write

$$N_0 = N_0(N, V, T) \quad (2.27)$$

The important point that we wanted to stress in this section is that condensation phenomena can be reduced to a variational problem. In the next section we will be more precise on which functional one has to find the extrema. More specifically, we will show that $\Omega^0(T, V, \mu, \psi_0)$ is related to the generating functional of the vertex functions computed at zero momentum. This provides a definite scheme for calculations in finite temperature field theory.

III. VARIATIONAL APPROACH

In order to study Bose–Einstein condensation we shall employ the so called variational method. As pointed out by L. Michel⁽⁸⁾ the formulation of this problem starts with the construction of a functional Γ (that for the breakdown of group symmetry is chosen to be group invariant) depending on a field φ and also on external parameters that we will label by T ($\Gamma(\varphi, T)$). Let φ_1 be a solution of the following variational problem

$$\left. \frac{\delta \Gamma(\varphi, T)}{\delta \varphi} \right|_{\varphi = \varphi_1} = 0 \quad (3.1)$$

The study of phase transitions can be pursued by analysing the behavior of the functional Γ for different field configurations φ_i 's^(10–12).

Within the canonical ensemble it is known that the functional to be extremized is the free energy of the system in the presence of a background field $\Gamma(\varphi, T)$ ^(11,12). In that context the study of phase transitions is carried out by analysing the difference between free energies associated to different backgrounds (one of them is taken to be the vacuum of the theory – φ_V) that is one studies the difference between free energies

$$F(\varphi_1, T) \equiv \Gamma(\varphi_1, T) - \Gamma(\varphi_V, T) \quad (3.2)$$

In fact there is plenty of space in the literature devoted to the study of the behaviour and the minima of the effective potential $V_{\text{eff}}(\varphi)$ ⁽¹⁰⁾. This thermodynamical potential is defined as

$$V_{\text{eff}}(\varphi) = \frac{T}{V} \left[\Gamma(\varphi) - \Gamma(\varphi_V) \right] \quad (3.3)$$

where φ is an uniform field configuration.

The idea is then, in close analogy with the phenomenological Landau's theory of phase transition⁽¹³⁾, to look for the extrema of $V_{\text{eff}}(\varphi)$. The minima corresponds to the order parameter. These minima varies with T . Assume that at some temperature T_0 the absolute minimum of V_{eff} is at φ_0 . Suppose now that at some temperature T_1 other minimum φ_1 becomes equivalent to that of φ_0 , that is $V_{\text{eff}}(\varphi_0, T_1) = V_{\text{eff}}(\varphi_1, T_1)$, and that for $T < T_1$ φ_1 becomes the new absolute minimum. In this case one has a phase transition with a discontinuous change in the order parameter.

We have shown in section II.2 that the problem of Bose–Einstein condensation can be formulated as a variational problem. In this section we will analyse a more general problem, that is, we will analyse the behaviour of the Gibbs potential when the system is in the presence of a non–uniform background field $\psi_b(\vec{x}, \tau)$. In other words, following ref.(10), let us analyse this more general problem and then, in the next section, consider the problem of uniform configurations that is relevant for studying Bose–Einstein condensation. $\Omega(\psi_b)$ represents in these circumstances the Gibbs potential associated to the system in the presence of the background field $\psi_b(\vec{x}, \tau)$. Let us look at the extrema of this functional, that is

$$\left. \frac{\delta \Omega}{\delta \psi_b} \right|_{\psi_b = \psi_c} = 0 \quad (3.4)$$

Let us consider the system described by the Hamiltonian (2.1) in the presence of a background field $\psi_b(\vec{x}, \tau)$. Under these circumstances one can introduce, in analogy with (2.20), the potential $\Omega(\psi_b)$ obtained from (2.5) through the substitution

$$\psi \longrightarrow \psi + \psi_b \quad (3.5)$$

That is,

$$\Xi(\psi_b) = e^{-\Omega(\psi_b)} = \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S(\psi + \psi_b)} \quad (3.6)$$

$\Omega(\psi_b)$ as defined in (3.6) is a functional of the background field ψ_b and it differs from the Gibbs potential Ω^0 defined in (2.7) by a factor β .

Let us analyse further the thermodynamical properties of the system in the presence of the background field ψ_b and of an external source J_0 . Under these circumstances, the relevant functionals are $\Xi(J, \psi_b)$ and $\tilde{\Omega}(J, \psi_b)$ defined by:

$$\Xi[J, \psi_b] = e^{-\tilde{\Omega}(J, \psi_b)} = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-S[\psi + \psi_b] - \int_0^\beta d\tau \int d^3x [J^*(x)\psi(x) + J(x)\psi^*(x)]} \quad (3.7)$$

By means of a change of variables one can write

$$\begin{aligned} \Xi[J, \psi_b] &= \Xi[J] e^{\int_0^\beta d\tau \int d^3x [J^*(x)\psi_b(x) + J(x)\psi_b^*(x)]} \\ &= e^{-\Omega(J)} e^{\int_0^\beta d\tau \int d^3x [J^*(x)\psi_b(x) + J(x)\psi_b^*(x)]} \end{aligned} \quad (3.8)$$

where $\Xi[J] = \Xi(J, 0)$ and $\Omega(J) = \tilde{\Omega}(J, 0)$ stands for the above functionals evaluated without the background field. As it is well known $\Xi(J)$ and $\Omega(J)$ are the functional generators of the disconnected and connected Green's functions, respectively. $\Xi(J, \psi_b)$ and $\tilde{\Omega}(J, \psi_b)$ stands for the same functionals in the presence of the background ψ_b .

From (3.7) and (3.8) it follows that

$$\tilde{\Omega}[J, \psi_b] = \Omega[J] - \int_0^\beta d\tau \int d^3x (J^*(x)\psi_b(x) + J(x)\psi_b^*(x)) \quad (3.9)$$

Being $\bar{\psi} = \frac{\delta \tilde{\Omega}[J, \psi_b]}{\delta J^*}$, the expected value of the field in the presence of J and ψ_b , we can write

$$\tilde{\Gamma}[\bar{\psi}, \psi_b] \equiv \tilde{\Omega}[J, \psi_b] - \int_0^\beta d\tau \int d^3x (\bar{\psi}^*(x)J(x) + \psi(x)J^*(x)) \quad (3.10)$$

where $\tilde{\Gamma}[\bar{\psi}, \psi_b]$ is the background field effective action.

By substituting (3.9) into (3.10) it follows that

$$\tilde{\Gamma}[\bar{\psi}, \psi_b] \equiv \Omega[J] - \beta^{-1} \int d\tau \int d^3x [J^*(x)(\bar{\psi}(x) + \psi_b(x)) + J(x)(\bar{\psi}^*(x) + \psi_b^*(x))] \quad (3.11)$$

consequently if one derives (3.8) with regard to J^* one obtains

$$\frac{\delta \Omega}{\delta J^*} = \bar{\psi} + \psi_b \quad (3.12)$$

From (3.12) one gets the following relationship

$$\bar{\psi} = \bar{\psi} - \psi_b \quad (3.13)$$

If one substitutes (3.13) into (3.11) one then gets the following relations:

$$\tilde{\Gamma}[\bar{\psi}, \psi_b] = \Omega[J] - \int_0^\beta d\tau \int d^3x (J^*(x)\bar{\psi}(x) + J(x)\bar{\psi}^*(x)) \equiv \Gamma(\bar{\psi}) \equiv \Gamma[\bar{\psi} + \psi_b] \quad (3.14)$$

Expression (3.14) is well known within the context of the background field method⁽¹⁴⁾. That is, the generating functional for the theory in the presence of the background can be obtained from the generating functional without the background field computed just by making the replacement $\bar{\psi} \rightarrow \bar{\psi} + \psi_b$.

The free energy in the presence of the background field is

$$\bar{\Omega}(\beta, \psi_b) \equiv \lim_{J \rightarrow 0} \bar{\Omega}[J, \psi_b] \equiv \lim_{J \rightarrow 0} \left[\bar{\Gamma}[\bar{\psi}, \psi_b] + \int_0^\beta d\tau \int d^3x (J^*(x) \psi(x) + J(x) \psi^*(x)) \right] \quad (3.15)$$

Finally, one notes that if ψ_b is a particular solution of the classical equation

$$\left. \frac{\delta \Gamma}{\delta \psi} \right|_{\psi = \psi_c} = 0 = J \quad (3.16)$$

that is

$$\psi_b = \psi_c = \bar{\psi} \quad (3.17)$$

then in the limit $J \rightarrow 0$ (3.13) leads to $\bar{\psi} = 0$. Under this circumstance it follows from (3.14) and (3.15) that

$$\Omega(\psi_c) = \Gamma[\psi_c] \quad (3.18)$$

Expression (3.14) is a very relevant one in our approach since it reduces the problem of determining the extrema of the Gibbs potential to that of finding the extrema of Γ . The method that we have used in order to show the equivalence between the Gibbs potential in the presence of an external field and the generating functional is not new and is known as the Background Field Method⁽¹⁴⁾.

The replacement $\psi \rightarrow \psi + \psi_b$ is clearly analogous to Bogoliubov's approach to the condensation phenomena. Our reading of Bogoliubov's method is that it allows to isolate the relevant degrees of freedom (the $\bar{p} = 0$ state) of the problem. One then proceeds to the minimization of the Gibbs potential as a function of this degree of freedom. The system will choose the configuration for which $\Omega(\psi_b)$ is an extremum.

IV. GENERAL THEORY OF BOSE-EINSTEIN CONDENSATION

Let us consider the thermodynamic properties of the interacting Bose gas described by (2.1) in the presence of a uniform background field that we assume to be real and positive. That is, one writes

$$\psi_b = \sqrt{\rho_b} \quad (4.1)$$

For such a background one can write, on general grounds

$$\Gamma = \Gamma(\rho_b, T) = V\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \bar{\Gamma}^{(n,m)}(0 \cdots 0, 0 \cdots 0, T) \rho_b^{\frac{n+m}{2}} \quad (4.2)$$

where $\bar{\Gamma}^{(n,m)}$ stands for the Fourier transform of the one-particle irreducible Green's function of the theory taken at zero momenta

$$\bar{\Gamma}^{(n,m)}(0 \cdots 0) = \bar{\Gamma}^{(n,m)}(p_1 \cdots p_n, \cdots p_m) \Big|_{p_1=p_2=\cdots=0} \quad (4.3)$$

The volume term, V in (4.2), comes from δ function involving momentum conservation whereas β comes from energy conservation.

If one defines Γ_{eff} as

$$\Gamma_{\text{eff}}(\rho_b, T) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \rho_b^{\frac{m+n}{2}} \bar{\Gamma}^{(n,m)}(0,0,0,0 \cdots 0, T) \quad (4.4)$$

then, from (3.8) and (4.2), it follows that for uniform configurations one can write

$$\Omega(\rho_b) = \beta \Omega^0(\rho_b, T) = \beta V \Gamma_{\text{eff}}(\rho_b, T) \quad (4.5)$$

We are now ready to establish the framework for studying Bose-Einstein condensation in field theory. In this context all one has to do is to write the thermodynamical variables in terms of $\Gamma_{\text{eff}}(\rho_b, T)$. The equation for determining the number of particles in the $\mathfrak{p} = 0$ state, for example, is equivalent to the equation

$$\left. \frac{d\Gamma_{\text{eff}}(\rho_b, T)}{d\rho_b} \right|_{\rho_c} = 0 \quad (4.6)$$

The solution of (4.6) leads to the determination of ρ_c as a function of μ and T ,

that is

$$\rho_c = \rho_c(T, \mu) \quad (4.7)$$

The dependence of the condensate on T and μ requires some care in order to determine the thermodynamical variables in the condensate phase. One needs to derive Ω with regard to the proper variables first and then substitute the condensate configuration ρ_c .

The equation of state, for instance, in the presence of the condensate should be inferred from the following equations

$$P = \Gamma_{\text{eff}}(\rho_c, \mu, T) \quad (4.8)$$

$$\frac{N}{V} = \left[\frac{\partial \Gamma_{\text{eff}}(\rho_b, T)}{\partial \mu} \right]_{\rho_b = \rho_c} \quad (4.9)$$

We say that there is Bose-Einstein condensate whenever there is a solution for the

equations (4.6)–(4.9) satisfying the condition

$$\rho_c > 0 \quad (4.10)$$

From equations (4.6) and (4.9) one should get the expression for the occupation of the $\bar{p} = 0$ state as a function of $\frac{N}{V}$ and T . That is

$$\rho_c = \rho_c \left[\frac{N}{V}, T \right] \quad (4.11)$$

The system is supposed to exhibit two phases. In the non condensed phase

$$\rho_c = 0 \quad (4.12)$$

whereas in the condensed phase:

$$\rho_c = \rho_c \left[\frac{N}{V}, T \right] \quad (4.13)$$

The critical temperature is the one that distinguishes the two phases. ρ_c plays the role of the order parameter. One defines the critical temperature as the one for which,

$$\rho_c \left[\frac{N}{V}, T_c \right] = 0 \quad (4.14)$$

The whole thermodynamics can then be inferred from equations (2.8)–(2.12) with all variables computed at the field theoretical configuration that makes Γ an extremum. As far as condensation is concerned $\Gamma_{\text{eff}}(\rho, T)$ is the relevant potential. In terms of $\Gamma_{\text{eff}}(\rho, T)$ we can write, in the condensed phase

$$P = \Gamma_{\text{eff}}(\rho_c, T) \quad (4.15)$$

$$\frac{N}{V} = \frac{\partial \Gamma_{\text{eff}}(\rho_b, T)}{\partial \mu} \Big|_{\rho_b = \rho_c} \quad (4.16)$$

$$U = V \frac{\partial}{\partial \beta} (\beta \Gamma_{\text{eff}}(\rho_b, T)) \Big|_{\rho_b = \rho_c} \quad (4.17)$$

$$F = V \Gamma_{\text{eff}}(\rho_c, T) + \mu N \quad (4.18)$$

$$S = \beta^2 V \frac{\partial}{\partial \beta} (\Gamma_{\text{eff}}(\rho_b, T)) \Big|_{\rho_b = \rho_c} - \mu N \beta \quad (4.19)$$

whereas in the non-condensed phase one has for the same variables:

$$P = \Gamma_{\text{eff}}(0, T) \quad (4.20)$$

$$\frac{N}{V} = \frac{\partial \Gamma_{\text{eff}}(0, T)}{\partial \mu} \quad (4.21)$$

$$U = V \frac{\partial}{\partial \beta} (\beta \Gamma_{\text{eff}}(0, T)) \quad (4.22)$$

$$F = V \Gamma_{\text{eff}}(0, T) + \mu N \quad (4.23)$$

$$S = -\beta^2 V \frac{\partial}{\partial \beta} (\Gamma_{\text{eff}}(0, T)) - \mu N \beta \quad (4.23)$$

Equations (4.15)–(4.19) are then the basic set of equations describing the condensation phenomena.

Although the field theoretical approach have already been used to treat Bose-Einstein condensation our proposal differs from the others. We have stressed the need for finding the dependence of the Gibbs potential in condensate configurations and, from this, to derive the whole thermodynamics. Furthermore our treatment is very general and explicit results depends only on our ability to determine $\Gamma_{\text{eff}}(\rho, T)$.

V. OPERATOR APPROACH AND ZERO TEMPERATURE EQUATION OF STATE⁽³⁾

It is possible establish the equivalence between Bogoliubov's method and the one presented here by making use of the operator approach, that is, the separation (2.19) is implemented in the sense of operators. In this case we separate the operators a_0 and a_0^\dagger from ψ and ψ^\dagger thus we write

$$\begin{aligned}\tilde{\psi} &= \hat{\psi} + \frac{a_0}{\sqrt{V}} \\ \tilde{\psi}^\dagger &= \hat{\psi}^\dagger + \frac{a_0^\dagger}{\sqrt{V}}\end{aligned}\quad (5.1)$$

In order to see that the method here presented is parallel to Bogoliubov's we shall work within the usual Green's functions approach. The definition analogous to (2.5) is

$$\Xi = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{iS[\psi]} \equiv e^{+\Omega i} \quad (5.2)$$

where

$$\begin{aligned}S[\psi] &= \int d^4x \left[-i\psi^*(x) \frac{\partial}{\partial t} \psi(x) + \frac{(\vec{\nabla} \psi^*(x)) \cdot (\vec{\nabla} \psi(x))}{2m} - \mu \psi^*(x) \psi(x) + \right. \\ &\quad \left. + \int d^4x' \int d^4x'' \psi^*(x) \psi(x) V(x-x') \psi^*(x') \psi(x'') \right].\end{aligned}\quad (5.3)$$

The Green's functions in the operator approach are written as time ordered products (T) of fields

$$i G(x_1, \dots, x_n; y_1, \dots, y_n) = \langle T \{ \psi(x_1) \dots \psi(x_n) \psi^\dagger(y_1) \dots \psi^\dagger(y_n) S \} \rangle / \Xi \quad (5.4)$$

where \hat{S} is the S-matrix operator for this system, that is

$$\hat{S} = T \exp \left\{ -i \int \int dx_1 dx_2 \psi^\dagger(x_1) \psi(x_1) V(x_1 - x_2) \psi^\dagger(x_2) \psi(x_2) \right\} \quad (5.5)$$

The m particle Green's functions in the condensed phase is defined as the one with $2m$ products of $a_0(a_0^\dagger)$ in (5.4) after the separation (5.1):

$$i G_0(t_1, \dots, t_m; t_1', \dots, t_m') = \left\langle T \left[\frac{a_0(t_1)}{\sqrt{V}} \dots \frac{a_0(t_m)}{\sqrt{V}} \frac{a_0^\dagger(t_1')}{\sqrt{V}} \dots \frac{a_0^\dagger(t_m')}{\sqrt{V}} S \right] \right\rangle \frac{1}{\Xi} \quad (5.6)$$

In his relevant paper on the method of Green's functions in the condensed phase Beliaev obtained two important results for our purposes. The first result concerns the general structure of the Green's functions in the condensed phase. They are factorizable as products of factors $K(t_i)$

$$i G_0(t_1, \dots, t_m; t_1', \dots, t_m') = K(t_1) \dots K(t_m) \cdot K(t_1') \dots K(t_m') \quad (5.7)$$

where $K(t)$ and $K^*(t)$ satisfies the integral equations

$$K(t) = \bar{C} + \int dt' \theta(t-t') \frac{1}{V} \frac{\delta \Omega(KK^*)}{\delta K^*(t')} \quad (5.8)$$

$$K^*(t) = \bar{C}^* + \int dt' \theta(t-t') \frac{1}{V} \frac{\delta \Omega(KK^*)}{\delta K(t')}$$

with \bar{C} (and \bar{C}^*) defined by

$$\bar{C}^2 + \frac{\bar{C}}{V} \int \frac{\delta \Omega(KK^*)}{\delta K(t)} dt = \frac{N_0}{V} = \rho_0 \quad (5.9)$$

The second important result is that in this operator approach Ξ can be written as the exponential of a functional Ω of the variables K and K^+ under the form

$$\Xi = e^{i\Omega(K^*K)} \quad (5.10)$$

and the functional Ω can be expanded in a series totally equivalent to (2.18). This series is, following Beliaev notation⁽³⁾,

$$\Omega(K^*K) = V \sum_W \int dt_1 \dots dt_m dt_1' \dots dt_m' W^{(m)}(t_1, \dots, t_m, t_1', \dots, t_m') K^*(t_1) \dots K^*(t_m) K(t_1') \dots K(t_m') \quad (5.11)$$

where $W^{(m)}(t_1, \dots, t_m, t_1', \dots, t_m')$ have been identified by Beliaev as "certain vacuum loops with m pairs of incomplete vertices". These vacuum loops are, as shown here, the vertex functions of the theory.

Instead of analysing the structure of the one-particle Green's function, as was done by Beliaev, we prefer to take an alternative route. This will provide a way of showing the equivalence between our approach and the one that makes use of operators. The alternative route will be to concentrate in the study of $\Omega(K^*K)$. It is then clear that the condition analogous to (3.4) is

$$\frac{\delta \Omega}{\delta K} = 0 \quad (5.12)$$

From (5.8) and (5.9) it follows that K is time independent and has a form

analogous to (4.1), that is

$$K = \sqrt{\rho_0} \quad (5.13)$$

By substituting (5.13) into (5.11) and adopting our notation

$$\Omega(\rho_0) = \tau V \sum_m \sum_n \tilde{\Gamma}^{(m,n)}(0,0,0,\dots,0) \frac{\rho_0^{\frac{m+n}{2}}}{m! n!} \quad (5.14)$$

where τ is the total time interval (analogous to β in finite temperature). If one goes to the Euclidian version ($\tau \rightarrow i\beta$) one gets at zero temperature

$$\Omega^0(\rho_0) = V \sum_m \sum_n \tilde{\Gamma}^{(m,n)}(0,\dots,0, T=0) \frac{\rho_0^{\frac{m+n}{2}}}{m! n!} \quad (5.15)$$

The properties of the condensed phase can now be studied directly from $\Omega^0(\rho_0)$. This, as we shall see in the following, is a shorter route than the one proposed by Beliaev in which the Green's functions themselves plays the relevant role. The minimization condition (5.12) implies

$$\sum_{m=0} \sum_n \tilde{\Gamma}^{(m,n)}(0,\dots,0, T=0) \frac{\rho_0^{\frac{m+n}{2} - 1}}{m! n!} \frac{(m+n)}{2} = 0 \quad (5.16)$$

which allows for the determination of ρ_0 as function of μ

$$\rho_0 = \bar{\rho}_0(\mu) \quad (5.17)$$

equation (2.9) implies now

$$\frac{N}{V} = \sum_m \sum_n \left[-\frac{\partial}{\partial \mu} \tilde{\Gamma}^{(m,n)}(0,\dots,0, T=0) \right] \frac{\rho_0^{\frac{m+n}{2}}}{m! n!} \quad (5.18)$$

Equations (5.16) and (5.18) allows us to determine ρ_0 as function of $\left[\frac{N}{V} \right]$

$$\rho_0 = \rho_0 \left[\frac{N}{V} \right] \quad (5.19)$$

the equation of state should be inferred from

$$P = - \sum_m \sum_n \left[\tilde{\Gamma}^{(m,n)}(0,\dots,0, T=0) \right] \frac{\rho_0^{\frac{m+n}{2}}}{m! n!} \quad (5.20)$$

and equations (5.16) and (5.18).

We shall see later that equation (5.16) that allows us to determine $\rho_0 = \rho_0(\mu)$ is equivalent to the one obtained by Beliaev by using the operator approach.

The relevance of our equations is that they determine the equation of state for helium superfluid for temperatures very close to zero. We shall write down this equation of state explicitly in the low density approximation in the next section.

Equation (5.18) will permit us to study the depletion of the condensate as a result of quantum effects^(15,16).

VI. SIMPLE EXAMPLES

VI.1. CLASSICAL APPROXIMATION OR ZERO-LOOP APPROXIMATION

We shall see that depending on the nature of the potential there can be Bose-Einstein condensation at the zero-loop level. In order to see this let us apply our scheme to an arbitrary potential $V(\vec{x}-\vec{x}')$ and look at the condition that the potential should satisfy in order that, at the classical level, there be Bose-Einstein condensation.

Let us denote by $\Gamma^{(0)}$ the zero-loop graphs contributing to this approximation. One has, schematically,

$$\Gamma^{(0)}(1,1) = \text{---/---}$$

$$\Gamma^{(0)}(2,2) = \text{---/---} \bullet \text{---/---}$$

where the blob, in this case, takes into account the non-locality of the interaction.

From (2.18), one can write for r independent configurations:

$$\Gamma^{(0)}(\psi) = \beta \int d^3\vec{x} \left[+ \frac{(\vec{\nabla}\psi)^*(\vec{\nabla}\psi)}{2m} - \mu \psi^* \psi \right] +$$

$$+ \beta \int \int d^3\vec{x} d^3\vec{x}' \psi(\vec{x}) \psi^*(\vec{x}') V(\vec{x}-\vec{x}') \psi^*(\vec{x}') \psi(\vec{x}) \quad (6.1)$$

The Euler Lagrange equations (3.1) are, in zero loop (or classical level) approximation

$$-\frac{\vec{\nabla}^2}{2m} \psi(\vec{x}) - \mu \psi + 2 \int d^3\vec{x}' V(\vec{x}-\vec{x}') \psi^*(\vec{x}') \psi(\vec{x}') \psi(\vec{x}) = 0 \quad (6.2)$$

For a constant background field one can write, from (6.1)

$$\Gamma^{(0)}(\rho_b) = \beta V \left[-\mu \rho_b + \varepsilon \rho_b^2 \right] \equiv \beta V \left[\Gamma^{\text{eff}}(\rho_b, T) \right] \quad (6.3)$$

where ε in (6.3) is the integral over the whole space of the potential

$$\varepsilon = \int d^3\vec{R} V(\vec{R}) \quad (6.4)$$

The Gibbs potential is then,

$$\Omega^0(\rho_b) = V \left[-\mu \rho_b + \varepsilon \rho_b^2 \right] \quad (6.5)$$

The condition that Ω^0 be minimum,

$$\left. \frac{d\Omega^0}{d\rho_b} \right|_{\rho_c} = 0 \quad (6.6)$$

leads to the condition

$$2\varepsilon \rho_c = \mu \quad (6.7)$$

whereas equations (4.9) and (5.5) implies

$$\rho_c = \frac{N}{V} \quad (6.8)$$

This means that at the zero-loop level (classical level) all particles are in the condensate. This approximation will then be useful in the description of the system at zero temperature and at low densities⁽¹⁶⁾.

The internal energy U is, by using (4.17) and (6.3)

$$U = \varepsilon V \left[\frac{N}{V} \right]^2 \quad (6.9)$$

whereas the pressure can be obtained from (4.15), (6.7) and (6.8). We get

$$P = 2\varepsilon \left[\frac{N}{V} \right]^2 \quad (6.10)$$

One notes from (6.10) that in order that the pressure be positive one has to have

$$\varepsilon > 0 \quad (6.11)$$

Condition (6.11) has been emphasized by Bogoliubov who calls it the "condition of thermodynamic stability of a gas at absolute zero". This condition imposes a restriction on the possible types of interacting forces that allows for Bose-Einstein condensation. Condition (6.11) follows also from the requirement that Ω^0 be minimum, that is

$$\frac{d^2 \Omega^0}{d\rho^2} > 0 \quad (6.12)$$

The equation of state (6.10) has been derived in ref. (14) for a hard sphere gas. Equations (6.7) and (6.10) can be found in some papers as well in text books⁽⁶⁾. We just wanted to show how one can derive them by using our general scheme.

At the zero loop level there is only one phase (the condensed one). Furthermore this occurs only for potentials satisfying (6.11). In particular, there is no condensation, at this level, for free particles. There is condensation of free particles, however, when one introduces quantum corrections. This will be done next.

VI.2. QUANTUM CORRECTIONS - FREE FIELDS

In order to illustrate how the method here proposed works, let us compute $\Omega(\rho_0, T)$ explicitly in the case of free fields.

For free fields, the integral representation (2.5) is reduced to a quadratic one, namely

$$\Xi = \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-\int_0^\beta \int d^3x \left[\psi^* \frac{\partial}{\partial \tau} \psi - \mu \psi^* \psi + \frac{\vec{\nabla} \psi^* \cdot \vec{\nabla} \psi}{2m} \right]} \quad (6.13)$$

By making the substitution (3.5) with ψ_b given by (4.1) one gets

$$\Xi(\rho_b) = e^{-\Omega(\rho_b)} = e^{V\beta\rho_b} \cdot \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-\int_0^\beta \int d^3x \left[\psi^* \frac{\partial}{\partial \tau} \psi - \mu \psi^* \psi + \frac{\vec{\nabla} \psi^* \cdot \vec{\nabla} \psi}{2m} \right]} \quad (6.14)$$

Performing now the quadratic integration one gets

$$\begin{aligned} \Omega(\rho_b, T) &= \beta \Omega^0(\rho_0, T) \equiv \beta V \Gamma_{\text{eff}}(\rho_0, T) \\ &= \beta V \left[-\mu\rho_b + \frac{1}{\beta} \int \int \int \frac{d^3p}{(2\pi)^3} \ln \left[1 - e^{-\beta \left(\frac{p^2}{2m} - \mu \right)} \right] \right] \quad (6.15) \end{aligned}$$

The extremum condition (4.6) implies, from (6.15), that

$$\mu = 0 \quad (6.16)$$

In this case we have two phases. In the condensed phase $\mu = 0$, and by imposing this condition and using (4.15)–(4.17), one gets the usual results^(1,17)

$$P = -\frac{1}{\beta} \iiint \frac{d^3\vec{k}}{(2\pi)^3} \ln \left[1 - e^{-\frac{\beta\vec{p}^2}{2m}} \right] \quad (6.17)$$

$$\frac{N}{V} = \rho_c + \frac{1}{(2\pi)^3} \int d^3\vec{k} \frac{1}{e^{\beta\vec{p}^2/2m} - 1} \quad (6.18)$$

$$\frac{U}{V} = \iiint \frac{d^3\vec{k}}{(2\pi)^3} \frac{k^2/2m}{e^{\beta\vec{p}^2/2m} - 1} \quad (6.19)$$

Other thermodynamical variables can be easily written down. They are found in text books⁽¹⁷⁾ and can easily be inferred from equations (4.18)–(4.19) by taking $\mu = 0$.

The condensed phase occurs for $\rho_c > 0$. From (5.18) it follows that for free fields the dependence of ρ_c with the temperature is

$$\begin{aligned} \rho_c &= \frac{N}{V} - \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{1}{e^{\beta\vec{p}^2/2m} - 1} \equiv \frac{N}{V} - \left[\frac{m}{\pi\beta} \right]^3 \int \frac{d^3\vec{k}}{e^{k^2} - 1} \\ &= \frac{N}{V} - \left[\frac{mT}{2\pi} \right]^{3/2} \zeta(3/2) \end{aligned} \quad (6.20)$$

where ζ in (6.20) stands for the Riemann zeta functions $\zeta(x)$. The condition $\rho_c > 0$ implies that BE condensation occurs for $T > T_c$ where

$$T_c = \frac{2\pi}{m} \left[\frac{N}{V} \zeta(3/2) \right]^{2/3} \quad (6.21)$$

In the non-condensed phase ($\rho_c=0$) one gets the same thermodynamics of an ideal Bose gas. Under these circumstances all the thermodynamical properties are derived from the Gibbs potential

$$\Omega^0(T, \rho_0=0) = \frac{V}{\beta} \iiint \frac{d^3\vec{k}}{(2\pi)^3} \ln \left[1 - e^{-\beta(\frac{\vec{p}^2}{2m} - \mu)} \right] \quad (6.22)$$

VI.3. LOW-DENSITY APPROXIMATION

Let us consider the first two terms of the effective potential in the expansions in powers of ρ_b in (4.4)

$$\Gamma_{\text{eff}}^{\text{L.D.}}[\rho_b, T] = -\mu(T) \rho_b + \frac{\varepsilon(T)}{2} \rho_b^2 \quad (6.23)$$

Approximation (6.23) can be thought as a low-density approximation since it will be a good approximation for determining the properties of the system for low-densities of the $\vec{p} = 0$. The general structure for $\mu(T)$ and $\varepsilon(T)$, in perturbation theory, is

$$\begin{aligned} \mu(T) &= \mu + \text{diagram 1} + \text{diagram 2} + \dots \\ &+ \text{diagram 3} + \dots + \text{diagram 4} + \dots \\ &\equiv \mu + F(T, \mu) \end{aligned} \quad (6.24)$$

and

$$\begin{aligned} \varepsilon(T) &= \varepsilon + \text{[diagram 1]} + \text{[diagram 2]} + \dots \\ &+ \text{[diagram 3]} + \dots + \text{[diagram 4]} + \dots \\ &= \varepsilon + G(T, \mu) \end{aligned} \quad (6.25)$$

The Feynman rules for computing these graphs are the ones appropriate for finite temperature field theory⁽⁶⁾.

In the low density approximation one can see that our framework is similar to that of Ginzburg-Landau⁽¹⁸⁾. The parameters $\mu(T)$ and $\varepsilon(T)$ are here completely determined from the Green's functions of the theory. These can be determined in perturbative theory. From (4.6) and (6.23) we have

$$\rho_c(T, \mu) = \frac{\mu(T)}{\varepsilon(T)} = -\frac{\mu + F(T, \mu)}{\varepsilon + G(T, \mu)} \quad (6.26)$$

Within the low density approximation it is easy to get the whole thermodynamics. One writes, for example

$$\frac{N}{V} = \left[1 - \frac{dF}{d\mu} \right] \left[\frac{\mu(T)}{\varepsilon(T)} \right] - \frac{1}{2} \frac{d\varepsilon}{d\mu} \left[\frac{\mu(T)}{\varepsilon(T)} \right]^2 \quad (6.27)$$

The zero-temperature limit of the low density approximation for a hard-sphere gas

$$V(\vec{x} - \vec{x}') = \lambda \delta^3(\vec{x} - \vec{x}') \quad (6.28)$$

is fairly easy to predict. For V given by (6.28) the effective potential has the same form as the classical one:

$$\Gamma_{\text{eff}}^{\text{L.D.}}[\rho_b, 0] = -\mu_R \rho_b + \frac{\lambda_R}{2} \rho_b^2 \quad (6.29)$$

where μ_R, λ_R are the renormalized chemical potential, and coupling constant. Within the low density approximation there is no depletion of the $\vec{p} = 0$ state. All particles are in the condensate:

$$\rho_c = \frac{N}{V}$$

The other thermodynamical variables (P, U, F) are the ones predicted by the classical results (6.7)–(6.10) with ε replaced by λ_R and μ replaced by μ_R .

Although the low density approximation is apparently good for describing the phase transition and determining the critical temperature from the condition

$$\mu(T_c) = 0 \quad (6.30)$$

one runs into difficulties in imposing (6.30) for the description of the system at the critical point. For $T = T_c$, with T_c given by (6.30), one gets zero pressure and, by using (6.27), $\frac{N}{V} = 0$. Our conclusion is that, up to the terms that we have worked out explicitly, this approximation do not lead to a sensible method for describing the critical point. A better description of the phase transition is achieved by summing the series (4.4) (by using the loop expansion) and then making the whole thermodynamics⁽¹⁹⁾.

VI.4. FURTHER GENERAL RESULTS

Without making any explicit calculation for a realistic model (this will be done in a future publication⁽¹⁹⁾) we will go a little further in getting general results and compare some expressions of ours with those available in the literature.

By employing the loop expansion one can write, on general grounds,

$$\Gamma_{\text{eff}}(\rho_b, T) = \Gamma_{\text{eff}}^{(0)}(\rho_b, T) + \Gamma_{\text{eff}}^1(\rho_b, T) \quad (6.31)$$

here $\Gamma_{\text{eff}}^{(0)}(\rho_b, T)$ is the zero-loop contribution to $\Gamma_{\text{eff}}(\rho_b, T)$ and is given, from (6.3), by

$$\Gamma_{\text{eff}}^{(0)}(\rho_b, T) = -\mu\rho_b + \epsilon\rho_b^2 \quad (6.32)$$

and Γ^1 are the higher loop contributions to Γ_{eff} .

Equation (4.6) for Γ_{eff} given by (6.31) and using (6.32) is then equivalent to

$$\mu = \left[\frac{\partial}{\partial \rho_b} \Gamma_{\text{eff}}^1(\rho_b, T) \right]_{\rho_c} + \epsilon\rho_b \quad (6.33)$$

This result is equivalent to one obtained by Beliaev⁽³⁾. Since in ref. (3) there was no equation analogous to (4.16) Beliaev suggested that from equation (6.33) we should determine μ once ρ_c is given. We can see that this is not the case since, as explained before, we have to deal with (6.33) and (4.16).

For a hard sphere gas in which ultraviolet divergences appears, equation (6.33) gives the renormalization condition for the chemical potential. The renormalization condition is⁽⁵⁾

$$\mu_R = \left[\frac{\partial}{\partial \rho_b} \Gamma_{\text{eff}}(\rho_b, 0) \right]_{\rho_b=0} \quad (6.34)$$

Another interesting relation that can be inferred from our approach is that it allows us to write μ in terms of average in the ensemble of the interaction energy density. By making the substitution (3.5) in (2.5) and by deriving with regard to ρ_b we get

$$\frac{1}{\Xi} \frac{d\Xi(\rho_b)}{d\rho_b} \equiv \mu V \beta - \frac{1}{\Xi} \frac{\partial}{\partial \rho_b} \langle H_{\text{in}}(\psi + \psi_0) \rangle_{\Xi} \quad (6.35)$$

So that, for the condensate configurations it follows that

$$\beta V \mu = \frac{\partial}{\partial \rho_c} \langle \chi_{\text{int}} \rangle \quad (6.36)$$

where χ_{int} in (6.33) stands for

$$\chi_{\text{int}} = \int_0^\beta \int d^3\tilde{x} \int d^3\tilde{x}' \psi^*(\tilde{x}) \psi(\tilde{x}) V(\tilde{x} - \tilde{x}') \psi^*(\tilde{y}) \psi(\tilde{y}) \quad (6.37)$$

By using periodic boundary conditions one can write

$$\psi(\tilde{x}, \tau) = e^{\frac{2\pi i \tau}{\beta}} \psi(\tilde{x}) \quad (6.38)$$

So that, as a result of the periodic boundary conditions (6.36) is equivalent to

$$V\mu = \frac{\partial}{\partial \rho_c} \langle H_{\text{int}} \rangle \quad (6.39)$$

where H_{int} is the interaction energy.

Result (6.39) is an extension to finite temperatures of a well known result in the study of condensation⁽⁶⁾. Expression (6.39) implies that μ gives the rate at which the interaction energy of the system changes when condensation occurs. No surprise then, that $\mu = 0$ in the free field case.

VII. CONCLUSIONS

In this paper we have shown how field theory at finite temperatures provides a way of achieving a complete and general description to the phenomena of Bose-Einstein condensation. The relevant parameter (the density of the $\beta = 0$ state) as well as all relevant thermodynamical variables are completely determined, in field theory at finite temperatures, from the Green's functions in momentum space at zero-momenta. This would permit us to achieve a description of helium superfluid at very low temperatures from a microscopic theory.

The basis of our approach is Bogoliubov's separation of the relevant degree of freedom in the study of Bose-Einstein condensation. The relevant degree of freedom is the uniform configuration. The system prefers among the uniform configuration that for which the Gibbs potential is an extremum. The problem is then reduced to a variation problem. The most interesting aspect of our approach is that, since the Gibbs potential for an arbitrary background configuration can be related to the generating functional of the vertex functions at zero momenta, one has a very well established scheme for determining all relevant physical quantities for the condensation phenomena.

The implementation of our scheme for making realistic predictions relies on our ability to get sensible approximations for Γ . One possibility would be to use a more refined low density approximation (up to quadratic terms one gets nothing interesting) selecting terms larger than the quadratic ones and then use the perturbative method to compute the coefficients in the expansion.

The most attractive and tractable approximation seems to be the loop-expansion. In the case of a hard sphere gas the effective potential is known analytically up to the one loop approximation⁽⁵⁾. We have worked out this case explicitly and shown that this approach is, in fact, richer than other methods presented so far to treat this problem⁽¹⁹⁾.

The background field method is essential in order to relate the Gibbs potential (as a function of the density of the $\bar{p} = 0$ state $\rho_0 - \Omega^0(\rho_0)$) to the effective potential ($\Gamma^{\text{eff}}(\rho_0)$). The relationship

$$\Omega^0(\rho_0, T) = V \Gamma_{\text{eff}}(\rho_0, T)$$

between these thermodynamical potentials is an essential one in order to obtain the whole thermodynamics correctly and, as a consequence, how to infer the properties of the system from the effective potential.

The use of the effective potential in the study of helium 4 is not new. In fact, in the papers of ref. (5) Toyoda studied the λ transition within the hard sphere gas approximation. There are however, important differences between the approaches proposed in this paper and Toyoda's one. We have been mainly concerned with the determination of the density of the condensate and in the determination of the whole thermodynamics from the effective potential. Toyoda, on the other hand was mainly concerned with the critical temperature, defined by him as

$$\left. \frac{d\Gamma_{\text{eff}}}{d\rho} \right|_{\rho=0} = 0 \quad T = T_c$$

Even for the determination of the critical temperature we will get a result different from Toyoda's⁽¹⁹⁾ since the critical temperature for us is defined by $\rho_c(T_c) = 0$. One then has to solve, for determining the critical temperature, the set of equations (4.6) and (4.9).

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