UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-884

ANATOMY OF RELATIVISTIC MEAN FIELD APPROXIMATIONS

S. Cruz Barrios and M.C. Nemes
Instituto de Física, Universidade de São Paulo

Janeiro/1991

Anatomy of Relativistic Mean Field Approximations

S. Cruz Barrios and M.C. Nemes
Instituto de Física
Depto. de Física Matemática
Universidade de São Paulo. C.P. 20516 SP, BRASIL

1990

Abstract

In the present work we have set up a scheme to treat field theoretical lagrangians in the same bases of the well known nonrelativistic many body techniques. We show here that fermions and bosons can be treated quantum mechanically in a symmetric way and obtain results for the mean field appoximation.

1 Introduction

Treating strongly interacting systems via effective lagranians requires the use of nonpetubative methods. In this context a very important tool is the self consistent mean field approximation, well know in the various fields of applications, nuclear physics, solid state physics, statistical mechanics and so on. Most of the successful mean field apoximations in field theoretical models of inteacting fermions and bosons[2] assume that the mesonic degrees of freedom behave classically, while the fermions are treated quantum mechanically. The semiclassical limit of such systems is an attractive and recent research topic due to its connection to chaos[4]. At the same time it is wel known that a unique prescription in obtaining such limit is ill defined[5]. It has been recently shown[4,6,7] that the mathematically

rigorous classical analogous of a spin-boson system can only be obtained by using a symmetrical mean field wave function. Only then the quantum Heisenberg equations can be shown to correspond exactly to their classical counterpart. In the present contribution we show that a nonperturbative scheme can be set up to justify the results obtained in such theories and treats both fermions and bosons in a symmetric way, ie, as quantum fields. Even in systems composed of fermion field only (such as the Nambu-Jona-Lasinio model) the present approximation reveals interesting properties in terms of the current language of nonrelativistic many body physics. In particular for the NJL model[8] we desintangle the Hartree and Fock contributions showing explicitely a very delicate cancellation which leads to the result that in this model only the Hartree term survives.

The paper is organized as follows: in section 2 we treat the $\sigma-\omega$ model[2, 3] using a quantum representation for the fermion and meson fields. Next we obtain the corresponding mean field approximation and show that the well known results obtained by treating the mesonic field classically can be justified in this symmetric quantum approach. Section 3 analyses within the same scheme the mean field approximation for the NJL model[8] showing explicitely that only the Hartree term survives due to the cancelation of the Fock contribution. Conclusion follow in section 4.

2 First Example: the $\sigma-\omega$ model

The lagrangian density for the $\sigma-\omega$ model is given by:

$$\begin{aligned} L_{\sigma-\omega} &= \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v\phi_v^\mu) - (m - g_s\phi_s)]\psi \\ &+\frac{1}{2}(\partial_\mu\phi_s\partial^\mu\phi_s - m_s^2\phi^2) \\ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2\phi_{v_\mu}\phi_v^\mu. \end{aligned} \quad (1)$$

We consider an infinite system and use the following expansion for the fermionic and mesonic fields:

$$\psi = \sum_{k,l,s}[\frac{(2\pi)^3}{V}\frac{m}{E_o(\vec{k})}]^{1/2}u_l(\vec{k},s)c_l(\vec{k},s)exp\{-i(-)^l k.x\} \quad (2)$$

$$\phi_{M_i} = \sum_{p_i}[\frac{1}{\sqrt{2v\alpha_{p_i}}}][b^i_{p_i} exp\{ip_i.x\} + b^{i\dagger}_{p_i} exp\{-ip_i.x\}] \quad (3)$$

with $(l = 0, 1)$ and

$$u_o(\vec{k}, s) = \sqrt{\frac{E_o(\vec{k}) + m}{2m}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}.\vec{k}}{E_o(\vec{k})+m}\chi_s \end{pmatrix}, \quad (4)$$

$$u_1(\vec{k}, s) = \sqrt{\frac{E_o(\vec{k}) + m}{2m}} \begin{pmatrix} \frac{-\vec{\sigma}.\vec{k}}{E_o(\vec{k})+m}\chi_s \\ \chi_s \end{pmatrix} \quad (5)$$

the Dirac Spinors of the positive and negative energies respectivelly, with :

$$\begin{aligned} \bar{u_{l'}}(\vec{k}, s')u_l(\vec{k}, s) &= \{(-)^l\delta_{ll'} - \frac{\vec{\sigma}.\vec{k}}{m}(1 - \delta_{ll'})\}\delta_{ss'} \\ u^\dagger_{l'}(\vec{k}, s')u_l(\vec{k}, s) &= \frac{E_o(\vec{k}, s)}{m}\delta_{ll'}\delta_{ss'} \end{aligned} \quad (6)$$

In this reprensentation, the opertor, $c_0(\vec{k}, s)$ annihilates a fermion with $E_o(\vec{k}) > 0(particles)$, $c_1(\vec{k}, s)$ creates a fermion with $E_o(\vec{k}) < 0(antiparticles)$ and b^i are the boson operator where i can be equal $s \equiv \sigma$ and $v \equiv \omega$ meson, respectivelly. The $|vac>$ state for noninteracting fields is defined as: $c_o(\vec{k}, s)|vac>= c^\dagger_1(\vec{k}, s)|vac>= 0$.

The Hamiltonian is obtained through $H_{\sigma-\omega} = \int d^3x H(x)$, where we can identify the terms: $H_D + H^i_M + H^i_{int}$, with $i = s, v$, and

$$\begin{aligned} H_D &= \frac{(2\pi)^3}{V} \sum_{k,k';l,l'} \frac{m\delta(\vec{k} - \vec{k'})}{[E_o(\vec{k})E_o(\vec{k'})]^{1/2}} exp\{-i[(-)^{l'}k'^o - (-)^l k^o]x_o\} \\ &\quad \bar{u}_l(\vec{k}, s)(\vec{\gamma}.\vec{k'} + m)u_{l'}(\vec{k'}, s')c^\dagger_l(\vec{k}, s)c_{l'}(\vec{k'}, s') \end{aligned}$$

$$\begin{aligned} H^s_M &= \sum_{p_s}(p^o_s)^2 b^{s\dagger}(p_s)b^s(p_s) \\ H^v_M &= \sum_{p_v}(p^o_v)^2 b^{v\dagger}(p_v)b^v(p_v) \\ H^s_{int} &= -g_\sigma[\frac{(2\pi)^3}{V}] \sum_{k,k';l,l'} \frac{m}{[E_o(\vec{k})E_o(\vec{k'})]^{1/2}} exp\{-i[(-)^{l'}k'^o - (-)^l k^o]x_o\} \\ &\quad \bar{u}_l(\vec{k}, s)u_{l'}(\vec{k'}, s')c^\dagger_l(\vec{k}, s)c_{l'}(\vec{k'}, s') \\ &\quad \sum_{p_s}[b^s(\vec{p_s})\delta(\vec{k'} - \vec{k} - \vec{p_s})exp\{ip^o_s x_o\} + b^{s\dagger}(\vec{p_s})\delta(\vec{k} - \vec{k'} + \vec{p_s})exp\{-ip^o_s x_o\}] \\ H^v_{int} &= -g_\omega[\frac{(2\pi)^3}{V}] \sum_{k,k';l,l'} \frac{m}{[E_o(\vec{k})E_o(\vec{k'})]^{1/2}} exp\{-i[(-)^{l'}k'^o - (-)^l k^o]x_o\} \\ &\quad \bar{u}_l(\vec{k}, s)u_{l'}(\vec{k'}, s')c^\dagger_l(\vec{k}, s)c_{l'}(\vec{k'}, s') \\ &\quad \sum_{p_v}[b^v(\vec{p_v})\delta(\vec{k'} - \vec{k} - \vec{p_v})exp\{ip^o_v x_o\} + b^{v\dagger}(\vec{p_v})\delta(\vec{k} - \vec{k'} + \vec{p_v})exp\{-ip^o_v x_o\}] \end{aligned} \quad (7)$$

2.1 Symmetric Mean Field Approximation

We assume the ground state $|\Psi>$ as a direct product of the fermionic and mesonic ground states, i.e., $|\Psi>=|\Phi_F>\otimes|\Phi_{M_i}>$ and in particular $|\Phi_F>=\prod_{k\le k_F} c\dagger_0(\vec{k}, s)|vac>$. The MFA is obtained solving simultanioully the following equations:

$$\begin{aligned} \sum_i\{h^i_M|\Phi^i_M>\} &= (E - E_F)|\Phi^i_M> \quad (8) \\ \sum_i\{h_f|\Phi_F>\} &= (E - E^i_M)|\Phi_F> \quad (9) \end{aligned}$$

with :

$$\begin{aligned} h^i_M &= H^i_M + <\Phi_F|H^i_{int}|\Phi_F> \quad (10) \\ h_f &= H_D + <\Phi^i_M|H^i_{int}|\Phi^i_M) \quad (11) \\ E_F &= <\Phi_F|H_D|\Phi_F> \quad (12) \\ E^i_M &= <\Phi^i_M|H^i_M|\Phi_M> \quad (13) \end{aligned}$$

The self consistent solutions of (5,6) to, $i = s$ and v, leads to:

$$\begin{aligned} h_F &= \sum_{k;l,l'} \frac{(2\pi)^3}{V}\frac{m}{E_o(\vec{k})} exp\{-i[(-)^{l'} - (-)^l]k^o.x_o\} \\ &\quad \{\bar{u}(\vec{k}, s)(\vec{\gamma}.\vec{k} + m * +\gamma^o V_o)u_l(\vec{k}, s)c\dagger_l(\vec{k}, s)c_{l'}(\vec{k}, s) \end{aligned} \quad (14)$$

with:

$$m* = m - g_\sigma^2 \frac{\rho_s(m)}{4m_s^2} \quad (15)$$

$$V_\circ = -g_\omega^2 \frac{\rho_v}{4m_v^2} \quad (16)$$

Note that in the present approach the well known expession for the reduced mass $m*$ is a direct consequence of he self consistency which comes from the symmetric quantum teatment of both degrees of fredom.

3 Second Example: N-J-L Model

Here we used the same expansion for the fermion field and consider the mass m as a free parameter to be obtained from the condition that the ground state energy is a minimum. The NJL hamiltonian can be written as:

$$H_{NJL} = \sum_{\alpha,\beta} K_{\alpha,\beta} c\dagger_\alpha c_\beta + \sum_{\alpha,\beta,\gamma,\delta} (V^{(1)}_{\alpha\beta\delta,\gamma} + V^{(2)}_{\alpha\beta\delta\gamma}) c\dagger_\alpha c\dagger_\beta c_\gamma c_\delta \quad (17)$$

where the kinetic term is:

$$\sum_{\alpha,\beta} K_{\alpha,\beta} c\dagger_\alpha c_\beta = \sum_{kl;k'l'} \{(-)^l \frac{|\vec{k}|^2}{E_\circ(\vec{k})} \delta_{ll'} \delta_{k,k'} + \frac{m}{E_\circ(\vec{k})} \vec{\sigma}.\vec{k}(1-\delta_{l,l'})\}$$
$$c_l^\dagger(\vec{k},s) c_{l'}(\vec{k},s) exp\{-i[(-)^{l'} - (-)^l] k^\circ.x_\circ. \quad (18)$$

and the two-body potential is:

$$\sum_{\alpha,\beta,\gamma,\delta} V^{(1)}_{\alpha\beta\gamma\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta = -\frac{g^2}{2} \sum_{k_i l_i, k'_i l'_i} [\frac{(2\pi)^3}{V}]^2 \frac{m^2 \delta(\vec{k'_1} - \vec{k_1} + \vec{k'_2} - \vec{k_2})}{[E_\circ(\vec{k_1}) E_\circ(\vec{k_2}) E_\circ(\vec{k'_1}) E_\circ(\vec{k'_2})]^{1/2}}$$
$$[\bar{u}_{l_1}(\vec{k_1},s_1) u_{l'_1}(\vec{k'_1},s'_1)][\bar{u}_{l_2}(\vec{k_2},s_2) u_{l'_2}(\vec{k_2},s'_2)]$$
$$: c^\dagger_{l_1}(\vec{k_1},s_1) c_{l'_1}(\vec{k'_1},s'_1) c^\dagger_{l_2}(\vec{k_2},s_2) c_{l'_2}(\vec{k'_2},s'_2) :$$
$$exp\{-i[(-)^{l'_1} k'^\circ_1 - (-)^{l_1} k^\circ_1 + (-)^{l'_2} k'^\circ_2 - (-)^{l_2} k^\circ_2].x_\circ\} \quad (19)$$

$$\sum_{\alpha,\beta,\gamma,\delta} V^{(2)}_{\alpha\beta\gamma\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta = +\frac{g^2}{2} \sum_{k_i,k'_i;l_i,l'_i} [\frac{(2\pi)^3}{V}]^2 \frac{m^2 \delta(\vec{k'_1} - \vec{k_1} + \vec{k'_2} - \vec{k_2})}{[E_\circ(\vec{k_1}) E_\circ(\vec{k_2}) E_\circ(\vec{k'_1}) E_\circ(\vec{k'_2})]^{1/2}}$$
$$[\bar{u}_{l_1}(\vec{k_1},s_1) \gamma_5 u_{l'_1}(\vec{k'_1},s'_1)][\bar{u}_{l_2}(\vec{k_2},s_2) \gamma_5 [u_{l'_2}(\vec{k'_2},s'_2)$$
$$: c^\dagger_{l_1}(\vec{k_1},s_1) c_{l'_1}(\vec{k'_1},s'_1) c^\dagger_{l_2}(\vec{k_2},s_2) c_{l'_2}(\vec{k'_2},s'_2) :$$
$$exp\{-i[(-)^{l'_1} k'^\circ_1 - (-)^{l_1} k^\circ_1 + (-)^{l'_2} k'^\circ_2 - (-)^{l_2} k^\circ_2].x_\circ\} \quad (20)$$

3.1 Mean Field Approximation

One equivalent way to obtain the ground state which mininmizes the eneergy is to impose the condition $< N_i|[H_{NJL}, c^\dagger_\alpha c_\beta]|N_i >= 0$, where $|N_i >$ is the ground state which we take as:

$$|N_i> = \prod_{k \leq k_F} c\dagger_\circ(\vec{k},s) c_1(\vec{k},s)|vac>$$

$$with:$$

$$c_l(\vec{k},s) = \begin{matrix} \delta_{l0}\Theta(k_f - |\vec{k}|) \\ \delta_{l1}\Theta(|\vec{k}| - k_f) \end{matrix} \quad (21)$$

In the two body potential terms we observe that the Hartree contributions are propotional to:

$$V_h^{(1)} \propto [\bar{u}_\circ(\vec{k},s) u_\circ(\vec{k},s)][\bar{u}_\circ(\vec{k_\alpha},s) u_1(\vec{k_\alpha},s)] \quad (22)$$

$$V_h^{(2)} \propto [\bar{u}_\circ(\vec{k},s) \gamma_5 u_\circ(\vec{k},s)][\bar{u}_\circ(\vec{k_\alpha},s) \gamma_5 u_1(\vec{k_\alpha},s)] \quad (23)$$

and the Fock contributions are propotional to:

$$V_f^{(1)} \propto [\bar{u}_\circ(\vec{k_\alpha},s) u_\circ(\vec{k},s)][\bar{u}_\circ(\vec{k},s) u_1(\vec{k_\alpha},s)] \quad (24)$$

$$V_f^{(2)} \propto [\bar{u}_\circ(\vec{k_\alpha},s) \gamma_5 u_\circ(\vec{k},s)][\bar{u}_\circ(\vec{k},s) \gamma_5 u_1(\vec{k_\alpha},s)] \quad (25)$$

Putting both contributions together we can see that the only terms that survives are the Hartree terms because the Hartree contributions in $V^{(2)}$ is null due to the structure of the γ_5 metric and the Fock contribution from $V^{(1)}$ cancels that coming from $V^{(2)}$. Explicitely:

$$< N_i|[H_{NJL}, c^\dagger_\alpha c_\beta]|N_i > \; = \; T + V_h^{(1)} + V_h^{(2)} + V_f^{(1)} + V_f^{(2)} = 0 \tag{26}$$

with:

$$T = \frac{m}{E_o(\vec{k_\alpha})}\vec{\sigma}\vec{k_\alpha}\delta_{k_\alpha k_\beta} \tag{27}$$

$$V_h^{(1)} = \lim_{V\to\infty}\frac{g^2}{2}[\frac{(2\pi)^3}{V}]\sum_k \frac{m}{E_o(\vec{k})E_o(\vec{k_\alpha})}(-2\vec{\sigma}.\vec{k_\alpha})\Theta(k_f - |\vec{k}|)\delta_{k_\alpha k_\beta} \tag{28}$$

$$V_h^{(2)} = 0 \tag{29}$$

$$V_f^{(1)} = \lim_{V\to\infty}\frac{g^2}{2}[\frac{(2\pi)^3}{V}]\sum_k \frac{m}{E_o(\vec{k})E_o(\vec{k_\alpha})}\{\vec{\sigma}.\vec{k_\alpha} + \vec{\sigma}.\vec{k}\}\Theta(k_f - |\vec{k}|)\delta_{k_\alpha k_\beta} \tag{30}$$

$$V_f^{(2)} = \lim_{V\to\infty}\frac{g^2}{2}[\frac{(2\pi)^3}{V}]\sum_k \frac{m}{E_o(\vec{k})E_o(\vec{k_\alpha})}\{-\vec{\sigma}.\vec{k_\alpha} + \vec{\sigma}.\vec{k}\Theta(k_f - |\vec{k}|)\delta_{k_\alpha k_\beta} . \tag{31}$$

Note that the last term on the r.h.s. of eq.(30) and (31) vanishes due to thew fact that the integrand is odd. Therefore (30) and (31) cancell exactly. Using eqs. (27) -(31) in (26) we obtain the following relation for the mass parameter:

$$m = \lim_{V\to\infty}\frac{g^2}{2}\frac{(2\pi)^3}{V}\sum_k \frac{m}{E_o(\vec{k})}2\Theta(k_f - |\vec{k}|). \tag{32}$$

4 Conclusions

In the present work we have adapted the scheme of traditional nonrelativistic many body techniques to treat field theoretical lagrangians. We have shown that fermions and bosons can be treated in the same way and the well kown mean field results are reproduced. Moreover in nonelativistic nuclear physics the method has proven to the very useful in going beyond mean

field theories (i.e. to treat collisions[9,10] and extansions of the RPA approximatin and so on). Recently a similar approach has been developed for the $\lambda\phi^4$ theory in order to study the problem of initial conditions[11]. The extension of the interacting fermions and bosons would be an interesting and open research topic.

References

[1] B.D.Serot and J.D.Walecka; in Advances in Nuclear Physics, vol.16 eds. J.W. Negele and E.Vogt.
L.S.Celenza and C.M.Shakin; Relativistic Nuclear Physics. World Scientific Lecture Notes in Physics, Vol.2.

[2] J.D.Walecka; Ann. of Physics 83, (1974) 491

[3] L.S.Celenza and C.M.Shakin; Phys.RevC24, (1981) 2704.

[4] M.B.Cibils, et al.; J.Phys.A: Math. Gen23 (1990) 545.

[5] R.Graham and M.Hohnerbach; Z.Phys.B57 (1984) 233.

[6] K.Hepp and E.H.Lieb; Helv.Phys.Acta 46 (1973) 573.

[7] K.Hepp and E.H.Lieb; Springer Lectures Notes in Physics 38 (1975) 178.

[8] Y.Nambu and G.Jona-Lasinio; Phys.Rev.122 (1961) 345; and Phys. Rev. 124 (1961) 246.

[9] M.C.Nemes and A.F.R. de Toledo Piza; Phys.Rev. C27 (1983) 862: C31 (1983) 613.

[10] S.Cruz Barrios and A.F.R. de Toledo Piza; Phys. A159 (1989) 440.

[11] L.Chi.Yong and A.F.R.de Toledo Piza; Moderm Phys. Let. A5,20 (1990) 1605.