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M.A.M. de Aguiar and K. Furuya
Instituto de Física "Gleb Wataghin", Departamento de Física do Estado Sólido e Ciências dos Materiais
Universidade de Campinas
C.P. 6165, Campinas, SP, Brasil

M.C. Nemes
Instituto de Física, Universidade de São Paulo

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M.A.M. de Aguiar[1] and K. Furuya[1]
Instituto de Física "Gleb Wataghin"
Departamento de Física do Estado Sólido e Ciências de Materiais
Universidade de Campinas, c.p.6165, Campinas, SP, Brasil.

M.C. Nemes[1]
Instituto de Física,
Departamento de Física-Matemática
Universidade de São Paulo, c.p.20516, SP, Brasil.

Abstract

We construct the classical analogue of the phenomenon of superradiance in the zero temperature limit and show that a simple geometrical interpretation can be given in the integrable case. The nonintegrable case is also studied and in both cases we find bifurcation of equilibrium for the same parameter values where this phase transition is known do occur in the Thermodynamical context. The ground state of the system is also studied in the framework of a mean field approximation and a simple analytical expression obtained. A connection between the classical and quantum points of view is presented.

1 Introduction

The Dicke model of Superradiance[1] describes a system of N identical two-level atoms in a linear cavity of volume V interacting with an electromagnetic field. The separation between the atoms is assumed to be large enough so that their mutual interaction can be discarded. Dicke. however. realized that. because the atoms interact with the same radiation field. they should be treated as a single system. and not independently[1]. One of the most important properties of the Dicke model

[1]Partially supported by CNPq and FINEP.

is the presence of a second order phase transition from normal to superradiance in the thermodynamical limit where N and V$\longrightarrow\infty$ with N/V finite. This was first shown rigorously by Hepp and Lieb[2]. In particular they evaluated exactly the partition function and correlation function in this limit. The transition to the superradiant regime is found to occur for a critical temperture. To which is a function of the parameters in the model. In the superradiant phase ($T < T_c$) all thermodynamically relevant states are shown to be states with non vanishing mean photon number and excited atomic states. This phase transition is therefor usually interpreted as a quantum phenomenon.

The existence of a classical limit for the Hamiltonian of Dicke's model was also rigorously shown to exist and to be unique[3]. In the present paper construct the classical analogue of the superradiant phase transition at zero temperature both for the model considered by Hepp and Lieb[2] and for its extension which includes antiressonant terms. The classical problem is shown to present bifurcation of equilibrium points at the same parameter values where phase transition to superradiance occurs. The character of the bifurcation depends on whether the antiressonant interaction is present or not.

Furthermore we present an analytical description of the ground state of the model in the context of a mean field approximation. The ground state energy is compared with the exact one and shown to be in excellent agreement. The superradiant phase transition for the ground state has been numerically observed by Scharf[4]. Here we present a simplified and analytical version of the phenomenon. Moreover the connection between the classical and quantum points of view is clearly established.

In section 2 we briefly review the Dicke model and define its classical analogue. Section 3 is devoted to the study of the bifurcations of equilibria in the classical model. In section 4 the study of the ground state in a mean field approach is presented as well as its connection to the classical results. Some concluding remarks can be found in section 5.

2 The Dicke Model and the Classical Analogue

The Hamiltonian at the Dicke model for a single radiation mode ν interactive with N atoms is given by ($\hbar = c = 1$)

$$H = a^\dagger a + \sum_{j=1}^{N}\left[\frac{\epsilon}{2}\sigma_i^z + \frac{\lambda}{2\sqrt{N}}\left(a\sigma_j^+ + a^\dagger\sigma_j^-\right)\right] \tag{1}$$

where a and $a^\dagger$ are creation and annihilation operators for the field, ϵ is the energy difference between the two levels of the atoms, λ is the coupling parameter measured in units of the field energy ν, and $\sigma_j^\pm = \sigma_j^x \pm \imath\sigma_j^y$ with σ_j^x, σ_j^y and σ_j^z the usual-Pauli matrices for the jth atom.

Defining colletive spin operators by

$$
\begin{aligned}
J_z &= \frac{1}{2}\sum_{j=1}^{N}\sigma_j^z \\
J_\pm &= \frac{1}{2}\sum_{j=1}^{N}\sigma_j^\pm
\end{aligned}
\tag{2}
$$

and noticing that $N = 2J$, where J is the total spin,we rewrite (1) as

$$H = H_0 + H_I \tag{3}$$

with

$$
\begin{aligned}
H_0 &= a^\dagger a + \epsilon J_z \\
H_I &= \frac{\lambda}{\sqrt{2J}}\left(aJ_+ + a^\dagger J_-\right)
\end{aligned}
$$

The phenomenon of superradiance is usually studied in connection to the Hamiltonian (3) i.e. $\lambda' = 0$ with the radiation field treated classically[5]. We summarize the argument in what follows considereing the quantized radiation field (eq.(3)). The rate of spontaneous emission of radiation from the system in a transition from an initial state $|\psi_i\rangle$ to a final state $|\psi_f\rangle$ is proportional to

$$|\langle\psi_f|H_I|\psi_i\rangle|^2 = \frac{\lambda^2}{2J}\left|\langle\psi_f\left|J_+a + J_-a^\dagger\right|\psi_i\rangle\right|^2 \tag{4}$$

For simplicity we assume in this discussion $\epsilon = 1$. In order to calculate the above matrix element it is important to notice that the spin projection plus the number of photons is a conserved quantity, for $[H_0, H_I] = 0$. Considering an initial state with M excited atoms and n photons $|M,n\rangle$ and the corresponding final state $|M',n'\rangle$ we notice that the matrix element (4) introduces the following selection rules

$$
\begin{aligned}
\Delta M &= \pm 1 \\
\Delta n &= \pm 1
\end{aligned}
\tag{5}
$$

and $M + n = M' + n'$. We assume $|\psi_i\rangle = |\frac{N}{2} - n, n\rangle$ which corresponds to $\frac{N}{2} - n$ excited atoms and the corresponding photon number n. We get

$$|\langle\psi_f|H_I|\psi_i\rangle|^2 = \left(\frac{\lambda}{\sqrt{N}}\right)^2 N(2n+1)^2 \tag{6}$$

where $0 \le n \le \frac{N}{2}$. Notice now that the maximum value for the rate of spontaneous emission occurs for $n = \frac{N}{2}$ which corresponds to $M = 0$,

$$|\langle\psi_f|H_I|\psi_I\rangle|^2 \cong \left(\frac{\lambda}{\sqrt{N}}\right)^2 N^3 \quad \text{for large N.}$$

This corresponds the coherent emission as compared to the incoherent result $\left(\frac{\lambda}{\sqrt{N}}\right)^2 \cdot N$ which is obtained for the case where no photons are presented $n = 0$ and all atoms are excited.
Finally, we include in eq.(3) the antiressonant terms to get

$$H = a^\dagger a + \epsilon J_z + \frac{\lambda}{\sqrt{2J}}\left(aJ_+ + a^\dagger J_-\right) + \frac{\lambda'}{\sqrt{2J}}\left(a^\dagger J_+ + aJ_-\right). \tag{7}$$

The classical analogue to the above Hamiltonian is obtained via coherent states. We start by defining the normalized coherent state

$$|zw\rangle = |z\rangle \otimes |w\rangle \tag{8}$$

where

$$
\begin{aligned}
|z\rangle &= e^{z\bar{z}/2}\, e^{za^\dagger}\,|0\rangle \\
|w\rangle &= \frac{1}{(1+w\bar{w})^J}\, e^{wJ_+}\,|J,-J\rangle
\end{aligned}
\tag{9}
$$

and $|0\rangle$ and $|J,-J\rangle$ are the ground states of field and atoms respectively, such that

$$a|0\rangle = 0; \quad J_-|J,-J\rangle = 0 \tag{10}$$

The classical Hamiltonian is then defined as

$$
\begin{aligned}
H_{cl} &= \langle zw|H|zw\rangle \\
&= z\bar{z} - \epsilon J\left(\frac{1-w\bar{w}}{1+w\bar{w}}\right) + \frac{2\sqrt{2J}}{1+w\bar{w}}\left[\lambda(\bar{w}z + w\bar{z}) + \lambda'(wz + \bar{w}\bar{z})\right]
\end{aligned}
\tag{11}
$$

In terms of "action and angle" variables $\underset{\sim}{I}$, $\underset{\sim}{\theta}$ defined by

$$
\begin{aligned}
w &= \sqrt{\frac{J+I_1}{J-I_1}}\, e^{\imath\theta_1} \\
z &= \sqrt{I_2}\, e^{\imath\theta_2},
\end{aligned} \tag{12}
$$

eq.(11) reads

$$
H_{cl} = \epsilon I_1 + I_2 + \frac{2\sqrt{J^2 - I_1^2}\sqrt{I_2}}{\sqrt{2J}}\left[\lambda\cos(\theta_1 - \theta_2) + \lambda'\cos(\theta_1 + \theta_2)\right] \tag{13}
$$

Here, I_1 represents the classical projection of J_z, varying from $-J$ to $+J$, and I_2 the density of photons. Making a last transformation to cartezian coordinates,

$$
\begin{aligned}
q_1 &= \sqrt{2(J+I)}\,\sin\theta_1, \\
p_1 &= \sqrt{2(J+I_1)}\,\cos\theta_1, \\
q_2 &= \sqrt{2I_2}\,\sin\theta_2, \\
p_2 &= \sqrt{2I_2}\,\cos\theta_2,
\end{aligned} \tag{14}
$$

we arrive at

$$
H_{cl} = \epsilon H_1 + H_2 - \epsilon J + \frac{\sqrt{2J - H_1}}{\sqrt{2J}}\left(\lambda_+ p_1 p_2 + \lambda_- q_1 q_2\right) \tag{15}
$$

where

$$
\begin{aligned}
H_1 &= \frac{1}{2}\left(p_1^2 + q_1^2\right), \\
H_2 &= \frac{1}{2}\left(p_2^2 + q_2^2\right), \\
\lambda_\pm &= \lambda \pm \lambda'.
\end{aligned} \tag{16}
$$

Further details of these calculations can be found in ref.[6].

3 Bifurcations of Equilibria

Defining a four–vector X and the symplectic matrix Λ by

$$
X = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{17}
$$

Hamilton's equations can be written in the compact form

$$
\dot{X} = \Lambda\,\nabla H_{cl} \tag{18}
$$

The equilibrium points of eq.(18) are defined by the condition $\dot{X} = 0$, or $\nabla H_{cl} = 0$. Writing this explicitly gives

$$
\begin{aligned}
\dot{q}_1 &= -\epsilon p_1 - \frac{\lambda_+ p_2}{\sqrt{2J}}\sqrt{2J - H_1} \\
&\quad + \frac{p_1}{2\sqrt{2J}\sqrt{2J - H_1}}\left(\lambda_+ p_1 p_2 + \lambda_- q_1 q_2\right) = 0
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\dot{p}_1 &= \epsilon q_1 + \frac{\lambda_- q_2}{\sqrt{2J}}\sqrt{2J - H_1} \\
&\quad - \frac{q_1}{2\sqrt{2J}\sqrt{2J - H_1}}\left(\lambda_+ p_1 p_2 + \lambda_- q_1 q_2\right) = 0
\end{aligned} \tag{20}
$$

$$
\dot{q}_2 = -p_2 - \frac{\lambda_+ p_1}{\sqrt{2J}}\sqrt{2J - H_1} = 0 \tag{21}
$$

$$
\dot{p}_2 = q_2 + \frac{\lambda_- q_1}{\sqrt{2J}}\sqrt{2J - H_1} = 0 \tag{22}
$$

Solving (21) and (22) for p_2 and q_2 and substituting into (19) and (20) yields

$$
\begin{aligned}
p_1\left[4J\left(\epsilon - \lambda_+^2\right) + 2\lambda_+^2 p_1^2 + q_1^2\left(\lambda_+^2 + \lambda_-^2\right)\right] &= 0 \\
q_1\left[4J\left(\epsilon - \lambda_-^2\right) + 2\lambda_-^2 q_1^2 + p_1^2\left(\lambda_+^2 + \lambda_-^2\right)\right] &= 0
\end{aligned} \tag{23}
$$

We first assume that $\lambda' \neq 0$ and also that $\lambda > \lambda'$, so that $\lambda_+ > \lambda_-$. In this case, it is easy to check that eq.(23) plus (21) and (22) have the following solutions:

(A1) If $\lambda_+^2 < \epsilon$ (and therefore $\lambda_-^2 < \epsilon$)

$$
q_1 = p_1 = q_2 = p_2 = 0 \quad \text{(the origin)} \tag{24}
$$

(A2) If $\lambda_+ > \epsilon$ but $\lambda_- < \epsilon$
origin and

$$q_1 = q_2 = 0\ ; \quad p_1 = \pm\sqrt{\frac{2J(\lambda_1^2 - \epsilon)}{\lambda_+^2}}\ ; \quad p_2 = \mp\sqrt{\frac{J(\lambda_+^4 - \epsilon^2)}{\lambda_+^2}} \quad \text{(the p-root)} \tag{25}$$

(A3) If $\lambda_+^2 > \epsilon$ and $\lambda_-^2 > \epsilon$
origin,
p-root (as given above), and

$$p_1 = p_2 = 0\ ; \quad q_1 = \pm\sqrt{\frac{2J(\lambda_-^2 - \epsilon)}{\lambda_-^2}}\ ; \quad q_2 = \mp\sqrt{\frac{J(\lambda_-^4 - \epsilon^2)}{\lambda_-^2}} \quad \text{(the q-root)} \tag{26}$$

The stability of these solutions is given by the eigenvalues of the matrix

$$H''_{ij} = \frac{\partial^2 H}{\partial x_i \partial y_j} \tag{27}$$

calculated at each of these points:
origin

$$H'' = \begin{pmatrix} \epsilon & \lambda_+ & 0 & 0 \\ \lambda_+ & 1 & 0 & 0 \\ 0 & 0 & \epsilon & \lambda_- \\ 0 & 0 & \lambda_- & 1 \end{pmatrix} \tag{28}$$

$$\text{eigenvalues:} \quad \epsilon \pm \lambda_+\ ; \quad \epsilon \pm \lambda_- \tag{29}$$

$$det\, H'' = \left(\epsilon - \lambda_+^2\right)\left(\epsilon - \lambda_-^2\right) \tag{30}$$

p-root

$$H'' = \begin{pmatrix} \frac{\epsilon+\lambda_1^2}{2} & \frac{\lambda_-}{\lambda_+}\sqrt{\frac{\lambda_+^2+\epsilon}{2}} & 0 & 0 \\ \frac{\lambda_-}{\lambda_+}\sqrt{\frac{\lambda_+^2+\epsilon}{2}} & 1 & 0 & 0 \\ 0 & 0 & \frac{\lambda_+^4}{\lambda_+^2+\epsilon} & \epsilon\sqrt{\frac{2}{\lambda_+^2+\epsilon}} \\ 0 & 0 & \epsilon\sqrt{\frac{2}{\lambda_+^2+\epsilon}} & 1 \end{pmatrix} \tag{31}$$

$$\text{eigenvalues: roots of} \quad \begin{cases} \mu^2 - \frac{\mu}{2}\left(3\epsilon + \lambda_+^2\right) + \frac{\left(\epsilon+\lambda_+^2\right)\left(\lambda_+^2-\lambda_-^2\right)}{2\lambda_+^2} & = 0 \text{ and} \\ \mu^2 - \frac{\mu\left(3\lambda_+^2+\epsilon\right)}{\lambda_+^2+\epsilon} + 2\left(\lambda_+^2 - \epsilon\right) & = 0 \end{cases} \tag{32}$$

$$det\, H'' = \frac{\left(\lambda_+^2 - \lambda_-^2\right)\left(\lambda_+^4 - \epsilon^2\right)}{\lambda_+^2} \tag{33}$$

q-root

$$H'' = \begin{pmatrix} \frac{2\lambda_-^4}{\lambda_-^2+\epsilon} & \epsilon\sqrt{\frac{2}{\lambda_-^2+\epsilon}} & 0 & 0 \\ \epsilon\sqrt{\frac{2}{\lambda_-^2+\epsilon}} & 1 & 0 & 0 \\ 0 & 0 & \frac{\epsilon+\lambda_-^2}{2} & \frac{\lambda_+}{\lambda_-}\sqrt{\frac{\lambda_-^2+\epsilon}{2}} \\ o & 0 & \frac{\lambda_+}{\lambda_-}\sqrt{\frac{\lambda_-^2+\epsilon}{2}} & 1 \end{pmatrix} \tag{34}$$

$$\text{eigenvalues: roots of} \quad \begin{cases} \mu^2 - \frac{\mu}{2}\left(3\epsilon + \lambda_-^2\right) - \frac{\left(\epsilon+\lambda_-^2\right)\left(\lambda_+^2-\lambda_-^2\right)}{2\lambda_-^2} & = 0 \\ \mu^2 - \frac{\mu\left(3\lambda_-^2+\epsilon\right)}{\lambda_-+\epsilon} + 2\left(\lambda_-^2 - \epsilon\right) & = 0 \end{cases} \tag{35}$$

$$det\, H'' = -\frac{\left(\lambda_+^2 - \lambda_-^2\right)\left(\lambda_-^4 - \epsilon^2\right)}{\lambda_-^2} \tag{36}$$

The eigenvalues for the p-root can be easily shown to be positive if $\lambda_+^2 > \epsilon$ but those of the q-root are two positive and two negative, characterizing a saddle point.

Thus, to summarize, the origin is the only equilibrium point for $\lambda_+^2 < \epsilon$. For $\lambda_+^2 > \epsilon$ but $\lambda_- < \epsilon$, the two p-roots bifurcate from the origin as new minima, the origin becoming a saddle point. This is exactly the point were the phase transition to superradiance occurs. For $\lambda_-^2 > \epsilon$, the origin becomes a local maximum and the q-roots appear as saddle points. Since no new minima have been generated, no equivalent phase-transition occurs at this point.

The case $\lambda' = 0$ is very peculiar and deserves a separate analysis. In this case, the expression in brackets in equations (23) degenerate in a single one. Therefore, besides the origin we have

$$2J\left(\epsilon-\lambda^2\right)+\lambda^2\left(p_1^2+q_1^2\right)=0 \quad (37)$$

or

$$q_1^2=\frac{2J\left(\lambda^2-\epsilon\right)-p_1^2\lambda^2}{\lambda^2} \quad (38)$$

Therefore we must have

$$p_1^2<\frac{2J\left(\lambda'-\epsilon\right)}{\lambda^2}>0 \quad (39)$$

And again $\lambda^2>\epsilon$ for the solution to exist, and the phase transition occurs at the same point. But now we have a whole family of minima satisfying

$$\frac{q_1^2+p_1^2}{2}=J\left(1-\frac{\epsilon}{\lambda^2}\right)\equiv R_1^2 \quad (40)$$

Using equations (21) and (22), we can calculate

$$\frac{q_2^2+p_2^2}{2}\equiv R_2^2=\frac{J}{2}\left(1-\frac{\epsilon^2}{\lambda^4}\right) \quad (41)$$

Thus, $R_2^2=I_2$ (from eq.(14))gives a classical measure of the average density of photons in the system, and $R_1^2=J+I_1$ (from eq.(14)) gives a classical measure of the number of excited atoms in the system. It is easy to check that in general I_1+I_2 is a constant of motion for $\epsilon=1$. We are now in a position to make a complete analogy with the superradiant phenomena discussed in the beginning of this section. The transition to the superradiant state corresponds to the bifurcation from the origin to a circle of minima. Moreover the classical analogue of the maximum coherence quantum state corresponds to $I_1=0$ and R_1^2 half of this maximum value. This limit coincides with $\lambda>>\epsilon$ in eqs.(40) and (41). We have therefore maximum area in both degrees of freedom.

When λ' is switched on, only 4 points on the circle remain: the p–roots and the q–roots.

4 Mean Field Calculation of the Ground–State:

We start this section presenting a general framework for a mean field calculation suitable for two interacting systems and proceed to analyse the ground state properties of the Dicke model according.

4.1 Mean Field Approach

Let us consider a quantum system composed of two interacting subsystems, described by the Hamiltonian

$$\hat{H}=\hat{H}_1+\hat{H}_2+\hat{H}_{12} \quad (42)$$

where H_1 and H_2 corresponds to the free hamiltonians of each subsystem and H_{12} their interaction. We wish to calculate the ground state and its energy in the context of a mean field approximation. This can be accomplished by assuming the ground state to be a product wave function of the type

$$|\psi\rangle\equiv|\psi_1\rangle\otimes|\psi_2\rangle \quad (43)$$

which obeys the equation

$$\hat{H}|\psi\rangle=E|\psi\rangle \quad (44)$$

It is easy to check that inserting eq.(43) into (44) and projecting int $|\psi_1\rangle$ and $|\psi_2\rangle$ yields the folowing two coupled equations

$$\hat{h}_1|\psi_1\rangle=E_1|\psi_1\rangle \quad (45)$$

$$\hat{h}_2|\psi_2\rangle=E_2|\psi_2\rangle \quad (46)$$

where h_1 (h_2) is a function of $|\psi_2\rangle$ ($|\psi_1\rangle$),

$$\hat{h}_1=\hat{H}_1+\left\langle\psi_2\left|\hat{H}_{12}\right|\psi_2\right\rangle \quad (47)$$

$$\hat{h}_2=\hat{H}_2+\left\langle\psi_1\left|\hat{H}_{12}\right|\psi_1\right\rangle \quad (48)$$

and

$$E_1 = E-\left\langle\psi_2\left|\hat{H}_2\right|\psi_2\right\rangle$$

$$E_2 = E-\left\langle\psi_1\left|\hat{H}_1\right|\psi_1\right\rangle$$

The pair of equations (47), (48) should be solved self consistently

4.2 Solution for the Ground State of the Dicke Model

There are two types of solution. One which corresponds to the product wave functions of the noninteracting system and therefore zero average photon number and no excited atoms. The other one is the "condensed"solution which corresponds to the superradiant phase, when the coupling constants are chosen as discussed in the previous section. This will be shown in what follows.

The mean field equations (47), (48) are highly nonlinear and we shall solve them for the Dicke model with the following Ansatz

$$|\psi\rangle = |z\rangle \otimes |w\rangle \tag{49}$$

where the state $|z\rangle$ and $|w\rangle$ are given by (9)

The calculation of $\hat{h}_1$ and $\hat{h}_2$ is now straightforward

$$\hat{h}_1 = a^\dagger a + \frac{\lambda}{1+ww^*}(w^* a + wa) + \frac{\lambda'}{1+ww^*}\left(w^* a^\dagger + wa\right) \tag{50}$$

$$\hat{h}_2 = \epsilon j_z + \frac{\lambda}{\sqrt{2j}}(z^* j_- + z j_+) + \frac{\lambda'}{\sqrt{2j}}(z^* j_+ + z j_-) \tag{51}$$

We can now calculate $\hat{h}_1 |z\rangle$ and $\hat{h}_2 |w\rangle$,

$$\begin{aligned} \hat{h}_1 |z\rangle &= \frac{1}{1+ww^*}\left(\lambda w^* z + \lambda' w z^*\right)|v\rangle \\ &+ \left[z + \frac{1}{1+ww^*}\left(\lambda w + \lambda' w^*\right)\right] a^\dagger |v\rangle \end{aligned} \tag{52}$$

$$\begin{aligned} \hat{h}_2 |w\rangle &= \left(-\epsilon j + \frac{\lambda}{\sqrt{2j}} 2j z^* w + \frac{\lambda'}{\sqrt{2j}} 2j z w\right)|w\rangle \\ &+ \left[w + \frac{\lambda}{\sqrt{2j}} z + \frac{\lambda'}{\sqrt{2j}} z^* - \frac{\lambda}{\sqrt{2j}} z^* w^2 - \frac{\lambda'}{\sqrt{2j}} z w^2\right] j_+ |w\rangle \end{aligned} \tag{53}$$

and verify that our Ansatz is in fact a solution of the mean field equations provided the following conditions are satisfied (the second term on the r.h.s. of eqs.(52),(53) should be zero)

$$z + \frac{1}{1+ww^*}\left(\lambda w + \lambda' w^*\right) = 0 \tag{54}$$

$$w + \frac{\lambda}{\sqrt{2j}}\left(z - z^* w^2\right) + \frac{\lambda'}{\sqrt{2j}}\left(z^* - z w^2\right) = 0 \tag{55}$$

The above equations will determine z and w self consistently. They correspond precisely to the classical equations for equilibrium in the complex variables (eq.(11))

$$\dot{z} = 0 = -\imath \frac{\partial H_{cl}}{\partial z^*} \qquad \text{corresponds to eq.(54)}$$

$$\dot{w} = 0 = -\imath \frac{1}{\Omega}\frac{\partial H_{cl}}{\partial z^*} \qquad \text{corresponds to eq.(55)}$$

where $\Omega = \left(\frac{1-ww^*}{1+ww^*}\right)$

This means that the energy minima will be exactly those found in the previous section and the corresponding wave functions given by the coherent states. The approximation was checked by comparing the energy minimum thus obtained with the exact one[5] for $\lambda = 1$ and $\lambda' = 0.4$

$$E_{GS}^{exact} = -5.559543 \qquad E_{GS} = -5.557959$$

5 Conclusions

In the present contribuition we constructed the classical analogue of the Dicke model and studied its phase transition at zero temperature. The superradiant phase is shown to have a simple geometrical interpretation in the integrable case ($\lambda' = 0$): the mean photon density is associated with the geometrical area of the oscillator phase space, the average number of excited atoms given by the area in the corresponding phase space. The case $\lambda' \neq 0$ is also studied and the minima (in both cases) shown to exhibit bifurcation of equilibria for the same parameter values where phase transition occurs in the Thermodynamic limit.

Furthermore we obtain an analytical expression for the ground state of the system within the context of a mean field approch and obtain an excellent agreement for the ground state energy as compared to the exact one for given parameter values. A connection between the classical and quantum points of view is presented.

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