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RAINBOWS IN $K_{i\eta}(\xi)$

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ABSTRACT

The modified Bessel function of imaginary order $K_{i\eta}(\xi)$ is evaluated using the uniform approximation. It is found that $K_{i\eta}(\xi)$ represents basically a rainbow scattering problem. Numerical comparison between the uniform representation and the direct integration is made. Corrections to the uniform formula are assessed.

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1. Introduction

The modified Bessel function of imaginary order $K_{i\eta}(\xi)$ is discussed in most books and handbooks of special functions[1,2]. No tables, however are given. Numerical integration of the integral that represents this function is necessary. This procedure, however, could generate large errors, depending on the value of η and ξ. It is the purpose of this paper to perform a uniform approximation analysis that permits writing the function in terms of tabulated Airy function and its first derivative. In section 2 we present this analysis and derive formula for the corrections. In section 3 we present numerical comparison between the direct integration of the function and the uniform approximation. Finally in section 4 we discuss the general results obtained and present several concluding remarks. We have encountered this function in a recent investigation of the Coulomb dissociation of neutron–rich nuclei[3] and were quite marveled by its properties. In particular, we discovered that $K_{i\eta}(\xi)$ represents basically a rainbow scattering problem.

2. The Function $K_{i\eta}(\xi)$ and Its Uniform Representation

The integral representation of $K_{i\eta}(\xi)$ is given by in e.g., Abramowitz and Stegun[1] (pg 376),

$$K_{i\eta}(\xi) = \int_0^\infty e^{-\xi \cosh t} \cos \eta t \, dt \qquad \left(|\arg \xi| < \tfrac{\pi}{2}\right) . \qquad (1)$$

Now since the integrand is even in t, we may rewrite (1) in the following form

$$K_{i\eta}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\xi \cosh t}\, e^{i\eta t}\, dt \quad . \tag{2}$$

We now change $t \to t + i\pi/2$ and perform the integration along the line defined by $-\infty + i\pi < t < +\infty + i\pi/2$. This leaves the result enchanged since the integrand in an analytic function. Using the relation $\cosh(t+i\pi/2) = +\, i \sinh t$ we have

$$K_{i\eta}(\xi) = \frac{1}{2} e^{-\pi\eta/2}\int_{-\infty}^{\infty} e^{-i(\xi \sinh t - \eta t)}\, dt \quad . \tag{3}$$

Eq.(3) is our starting point for applying the uniform approximation. According to the usual procedure of Chester, Friedman and Ursell[4], we map the function $\xi \sinh x - \eta x$ into

$$\xi \sinh t - \eta t = \frac{\mu^3}{3} + x\mu \tag{4}$$

thus

$$K_{i\eta}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} e^{i(\mu^3/3 + x\mu)} \left[\frac{dt}{d\mu}\right] d\mu \quad . \tag{5}$$

Before we procede, we remark that the phase of the integrand in (2) is stationary when

$$\xi \cosh \pm t = \eta$$

or

$$t = \pm \cosh^{-1}\frac{\eta}{\xi} \qquad \eta > \xi \tag{6}$$

and

$$t = \pm i \cos\frac{\eta}{\xi} \qquad \eta < \xi \quad .$$

The situation can be easily understood from figure 1 where $\xi \cosh t$ is plotted vs. t. The minimum of $\xi \cosh t$ is at $t = 0$ and represents $\xi = \eta$, which is the "rainbow" η. For $\eta > \xi$ one is on the bright side of the rainbow whereas $\eta < \xi$ represents the dark side exemplified by two pure imaginary stationary points (complex conjugate of each other). When looked at from the cubic map, eq.(4), the stationary points are given by

$$\mu = \pm\sqrt{-x} \quad . \tag{7}$$

The function $dt/d\mu$ is, as usual, expanded as follows

$$\frac{dt}{d\mu} = \sum_{n=0}^{\infty} a_n(\mu^2 + x)^n \quad . \tag{8}$$

The coefficients in (8) are to be found by repeated differentiation of (4) at the stationary points (7). Thus, for a_0, a_1 and a_2, we need, $a_0 = (dt/d\mu)_{s.p.}$, $a_1 2\mu = (d^2t/d\mu^2)_{s.p.}$ and $2a_1 + 8\mu^2 a_2 = (d^3t/d\mu^3)_{s.p.}$. Accordingly

$$(\xi \cosh - \eta)\frac{dt}{d\mu} = \mu + x \tag{9.a}$$

$$\xi \sinh t \left[\frac{dt}{d\mu}\right]^2 + (\xi \cosh t - \eta)\frac{d^2t}{d\mu^2} = 2\mu \tag{9.b}$$

$$\xi \cosh t \left[\frac{dt}{d\mu}\right]^3 + 3\xi \sinh t \frac{dt}{d\mu}\frac{d^2t}{d\mu^2} + (\xi \cosh t - \eta)\frac{d^3t}{d\mu^3} = 2 \tag{9.c}$$

$$\xi \sinh t \left[\frac{dt}{d\mu}\right]^4 + 6\xi \cosh t \left[\frac{dt}{d\mu}\right]^2 \frac{d^2t}{d\mu^2} + 3\xi \sinh t \left[\frac{d^2t}{d\mu^2}\right]^2 + 4\xi \sinh t \frac{dt}{d\mu}\frac{d^3t}{d\mu^3} +$$

$$+ (\xi \cosh t - \eta)\frac{d^4t}{d\mu^4} = 0 \quad . \tag{9.d}$$

At the stationary points $\xi \cosh t - \eta = 0$ and $\mu^2 = -x$, and thus eq.(9.b) gives

$$\left[\frac{dt}{d\mu}\right]^2 = \frac{2\mu}{\xi \sinh t} = \frac{2\mu}{\xi\sqrt{\cosh^2 t-1}} = \frac{2\mu}{\xi\sqrt{\eta^2/\xi^2-1}} = \frac{2\sqrt{-x}}{\xi\sqrt{\eta^2-\xi^2}} = \left[\frac{-4x}{\eta^2-x^2}\right]^{1/2}$$

or

$$\left[\frac{dt}{d\mu}\right]_{s.p.} = \left[\frac{-4x}{\eta^2-x^2}\right]^{1/4} \equiv a_0 \quad . \tag{10}$$

Eqs. (9.c) and (9.d) give

$$a_1 = \frac{1}{2\mu}\left[\frac{d^2t}{d\mu^2}\right]^2_{s.p.} = \frac{2-4\left[\frac{-4x}{\eta^2-\xi^2}\right]^{3/4}}{6\sqrt{\eta^2-\xi^2}\left[\frac{-4x}{\eta^2-\xi^2}\right]^{1/4}(-x)^{1/2}} \tag{11}$$

and

$$a_2 = \frac{\left[\frac{d^3t}{d\mu^3}\right]_{s.p.} - 2a}{8(-x)} \text{ , with}$$

$$\left[\frac{d^3t}{d\mu^3}\right]_{s.p.} = \frac{-\sqrt{\eta^2-\xi^2}\left[\frac{dt}{d\mu}\right]^4_{s.p.} - 6\eta\left[\frac{dt}{d\mu}\right]^2_{s.p.}\left[\frac{d^2t}{d\mu^2}\right]^2_{s.p.} - 3\sqrt{\eta^2-\xi^2}\left[\frac{d^2t}{d\mu^2}\right]^2_{s.p.}}{4\sqrt{\eta^2-\xi^2}\left[\frac{dt}{d\mu}\right]_{s.p.}} \quad . \tag{12}$$

From eq.(4), we have for x that appear, in (10), (11) and (12), in the illuminated region, $\eta > \xi$

$$-\sqrt{\eta^2-\xi^2} + \eta\cosh^{-1}\frac{\eta}{\xi} = +\frac{2}{3}(-x)^{3/2} \quad . \tag{13}$$

In the forbidden region, $\eta < \xi$

$$\sqrt{\eta^2-\xi^2} - \eta\sin^{-1}\sqrt{1-\frac{\eta^2}{\xi^2}} = \frac{2}{3}x^{3/2} \quad . \tag{14}$$

Inserting (8) (kepping up to second–order terms) into (15), we have finally the desired formula, in terms of Airy's function $Ai(x)$ and its first derivative $A'i(x)$

$$K_{i\eta}(\xi) = \left[a_0\, Ai(x) - 2a_2\, A'i(x)\right] e^{-\pi\eta/2}\,\pi \quad , \tag{15}$$

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\mu^3/3+\mu x)}\, d\mu$$

where a_0 and a_2 are given by eqs. (10) and (12) respectively. Notice that the a_1 term given zero since it is odd. Higher order corrections can be easily generated (see Appendix I).

3. Rainbow in the Order

The discussion in the preceeding section can be made more transparent when compared with a scattering problem[5]. Here one speaks about the scattering at a given angle, which is the conjugate variable to the integration variable, the orbital angular momentum. The representation eq.(1) of $K_{i\eta}(\xi)$ identifies the order η with the "*angle*".

For a given value of η two values of t contributes, which are the two stationary points determined from the equation

$$\xi \cosh(\pm t) = \eta \quad (16)$$

where the function $\xi \cosh t$ represents a defflection function. The analogy with a scattering problem is a bit ill–based since both positive as well as negative t contribute. In a scattering situation only positive value of the angular momentum enter is the discussion. In any case $K_{i\eta}$ for a fixed value of ξ exhibit a double rainbow form as a function of η. The rainbow η is just $\eta = \xi$. For $\eta < \xi$ one is the shadow of the rainbow, where as $\eta > \xi$ represents the illuminated region (oscillatory behaviour). Both positive and negative values of η can be considered.

In Fig.(2) we present $K_{i\eta}(\xi)$ for $\xi = 0.1, 0.5, 1.0, 5.0$ and 10.0 vs. η. The rainbow in the order of $K_{i\eta}(\xi)$ is clearly exhibited.

4. Rainbow in the Argument

The analogy with scattering becomes more sound when discussing the behaviour of $K_{i\eta}(\xi)$ vs. ξ for a fixed η. To make full use of this analogy it is more convenient to use another representation for the function. This new representation is obtained from eq.(1) by a change of varibale $\sinh t = \lambda$. Then

$$K_{i\eta}(\xi) = \int \frac{d\lambda}{\sqrt{1+\lambda^2}}\, e^{i\eta \sinh^{-1}\lambda - i\lambda\xi} \;, \quad (17)$$

which when integrated by parts, yields

$$K_{i\eta}(\xi) = \xi \int dx\, e^{i\eta \sinh^{-1}\lambda - i\lambda\xi} \;. \quad (18)$$

The stationary points are determined from the condition

$$\eta \frac{d}{dx} \sinh^{-1}\lambda = \xi = \eta \frac{1}{\sqrt{\lambda^2+1}} \quad (19)$$

which is just eq.(6) rewritten with the new variale λ. The deflection function $\eta/\sqrt{1+\lambda^2}$ is plotted in fig.(3). Again the rainbow value of ξ is $\xi = \eta$. In figure (4) we show $K_{i0.1}(\xi)$, $K_{i1}(\xi)$, $K_{i5}(\xi)$ and $K_{i10}(\xi)$ which clearly show the Airy pattern that characterizes the rainbow in the argument.

5. Conclusions

In this paper the uniform approximation is used to express the function $K_{i\eta}(\xi)$ in terms of Airy's function and its first derivative. It is found that $K_{i\eta}(\xi)$ represents, as a functions of the order η and argument ξ, a rainbow scattering situation. Further, we verified that the uniform series, Eq.(15) is rapidly convergent. In fact a_2/a_0 is almost always less than 1%. At the rainbow, $\eta = \xi$, we find the relation $\frac{a_2}{a_0} = \frac{1}{2^{2/3}\cdot 70\cdot \eta^{4/3}}$ $\left[\text{or } \frac{1}{2^{2/3}\cdot 70\cdot \xi^{4/3}}\right]$, and thus, as long as $\eta(\xi)$ is not very small, the second term in (15) would contribute by, at most, a few %,s .

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1) M. Abramocoitz and I.A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, DC, 1964).

2) I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).

3) M.S. Hussein, M.P. Pato and C.A. Bertulani, submitted for publication.

4) C. Chester, B. Friedman and F. Ursell, *Proc. Camb. Philos. Soc.* **53** (1957) 599.

5) See, e.g., K.W. Ford and J.A. Wheeler, *Ann. Phys.* (N.Y.) **7** (1959) 259; **7** (1959) 287.

APPENDIX I

In this appendix we derive the necessary formula to calculate the higher–order corrections not taken into account in our calculation based on Eq.(15). To be specific, we calculate the coefficient a_0, a_1, a_2, a_3 and a_4. The first three coefficients have been already used.

From Eq.(15), up to a_4, we have

$$\frac{dt}{d\mu} = a_0 + a_1(\mu^2+x) + a_2(\mu^2+x)^2 + a_3(\mu^2+x)^3 + a_4(\mu^2+x)^4 \quad . \tag{I.1}$$

The following relations then follows at $\mu^2 = -x$.

$$\begin{aligned}
\frac{dt}{d\mu} &= a_0 \\
\frac{d^2t}{d\mu^2} &= 2\mu\, a_1 \\
\frac{d^3t}{d\mu^3} &= 2a_1 + 8\mu^2\, a_2 \\
\frac{d^4t}{d\mu^4} &= 24\mu\, a_2 + 48\mu^3\, a_3 \\
\frac{d^5t}{d\mu^5} &= 24\, a_2 + 284\mu^2\, a_3 + 384\mu^4\, a_4 \quad .
\end{aligned} \tag{I.2}$$

The systems of equations (I.2) relates the coefficients a_0, a_1, a_2, a_3, a_4 to the derivatives $\frac{dt}{d\mu}$, $\frac{d^2t}{d\mu^2}$, $\frac{d^3t}{d\mu^3}$, $\frac{d^4t}{d\mu^4}$ and $\frac{d^5t}{d\mu^5}$. These derivatives in turn are evaluated from the mapping relation, Eq.(4). We find the relations 9.b, 9.c, 9.d and

$$5\,\xi \sinh t \left[\frac{d^4t}{d\mu^4}\right]\left[\frac{dt}{d\mu}\right] + 10\,\xi \cosh t \left[\frac{d^3t}{d\mu^3}\right]\left[\frac{dt}{d\mu}\right]^2 + $$

$$+ 16\,\xi \sinh t \left[\frac{d^3t}{d\mu^3}\right]\left[\frac{d^2t}{d\mu^2}\right] + 15\,\xi \cosh t \left[\frac{d^2t}{d\mu^2}\right]^2\left[\frac{dt}{d\mu}\right] + \qquad \text{(I.3)}$$

$$+ 4\,\xi \sinh t \left[\frac{d^2t}{d\mu^2}\right]\left[\frac{dt}{d\mu}\right]^3 + \xi \cosh t \left[\frac{dt}{d\mu}\right]^5 = 0 \ .$$

Eq.(I.3) supplies a relation that relates the first, second, third and fourth derivative of t with respect to μ. For the obtention of the coefficient a_4, we need to have one more relation that involves the fifth derivative. We find

$$6\,\xi \sinh t \left[\frac{d^5t}{d\mu^5}\right]\left[\frac{dt}{d\mu}\right] + 15\,\xi \cosh t \left[\frac{d^4t}{d\mu^4}\right]\left[\frac{dt}{d\mu}\right]^2 + $$

$$+ 26\,\xi \sinh t \left[\frac{d^4t}{d\mu^4}\right]\left[\frac{d^2t}{d\mu^2}\right] + 66\,\xi \cosh t \left[\frac{d^3t}{d\mu^3}\right]\left[\frac{d^2t}{d\mu^2}\right]\left[\frac{dt}{d\mu}\right] + $$

$$+ 14\,\xi \sinh t \left[\frac{d^3t}{d\mu^3}\right]\left[\frac{dt}{d\mu}\right]^3 + 16\,\xi \sinh t \left[\frac{d^3t}{d\mu^3}\right]^2 + \qquad \text{(I.4)}$$

$$+ 15\,\xi \sinh t \left[\frac{d^2t}{d\mu^2}\right]^3 + 27\,\xi \sinh t \left[\frac{d^2t}{d\mu^2}\right]^2\left[\frac{dt}{d\mu}\right]^2 + $$

$$+ 9\,\xi \cosh t \left[\frac{d^2t}{d\mu^2}\right]\left[\frac{dt}{d\mu}\right]^4 + \xi \sinh t \left[\frac{dt}{d\mu}\right]^6 = 0 \ .$$

From Eq.(10) for $\left[\frac{dt}{d\mu}\right]$, one can then use Eqs.(11), (12), (I.2), (I.3) and (I.4), to obtain a_3

and a_4. Notice that taking terms up to a_4 in the evaluation of Eq.(5) we find

$$K_{i\eta}(\xi) = \pi\, e^{-\pi\eta/2}\left[\left(a_0 - 432x\, a_4\right) Ai(x) - 2a_2 A'i(x)\right] \ . \qquad \text{(I.5)}$$

The terms with a_1 and a_3 do not contribute. The next order contributing corrections proportional to a_6 would supply a term proportional to $A'i(x)$.

FIGURE CAPTIONS

Figure 1. The order "deflection function" $\xi \cosh t$ vs t for several values of ξ.

Figure 2. a) $K_{i\eta}(0.1)$ b) $K_{i\eta}(1)$ c) $K_{i\eta}(5)$ and d) $K_{i\eta}(10)$. Plotted is $e^{\pi\eta/2} K_{i\eta}(\xi)$.

Figure 3. The argument "deflection function" $\xi = \eta \frac{1}{\sqrt{1+\lambda^2}}$, for several values of η.

Figure 4. a) $K_{i0.1}(\xi)$, b) $K_{i1}(\xi)$, c) $K_{i5}(\xi)$ and d) $K_{i10}(\xi)$. Plotted is $e^{\pi\eta/2} K_{i\eta}(\xi)$. The insets in a) and b) are $K_{i\eta}(\xi)$ for very small ξ.

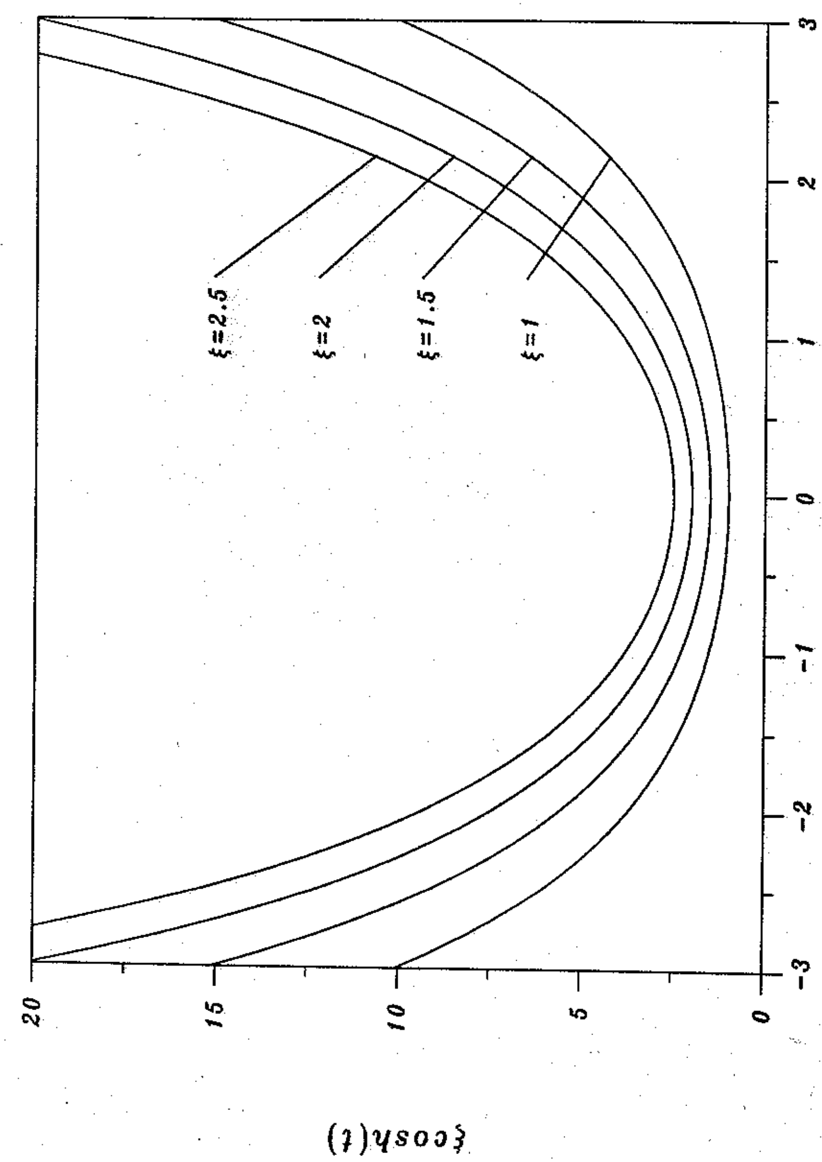
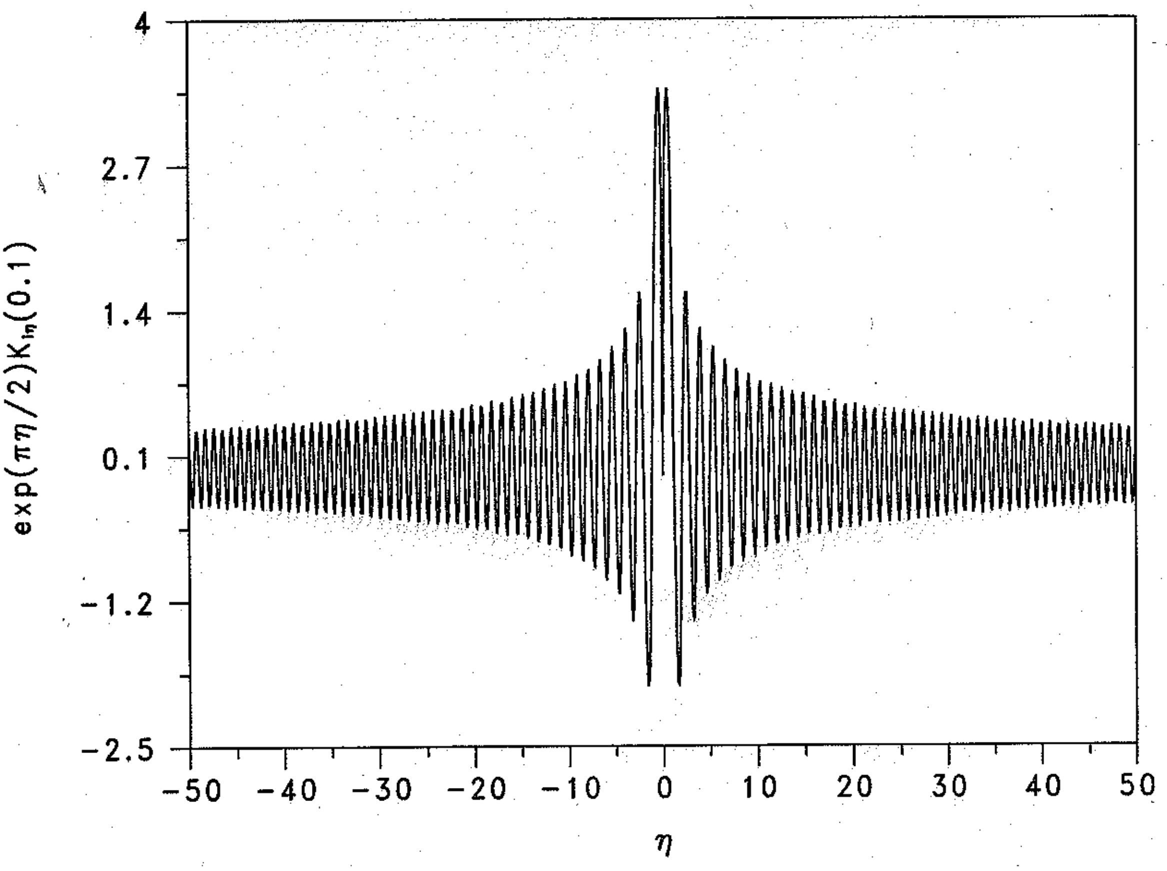
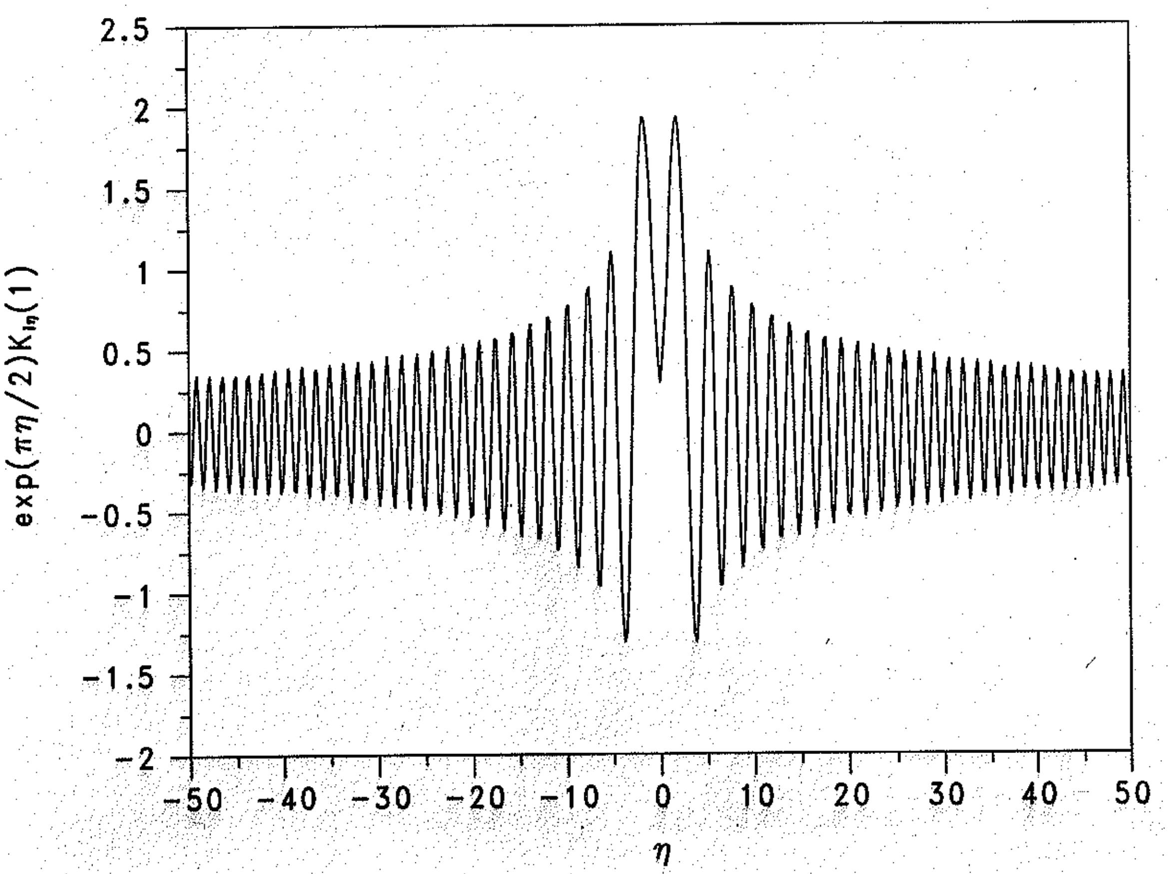


Fig. 1



4
2.7
1.4
0.1
-1.2
-2.5
exp(πη/2)K_iη(0.1)
-50 -40 -30 -20 -10 0 10 20 30 40 50
η

Fig. 2a



2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
exp(πη/2)K_iη(1)
-50 -40 -30 -20 -10 0 10 20 30 40 50
η

Fig. 2b

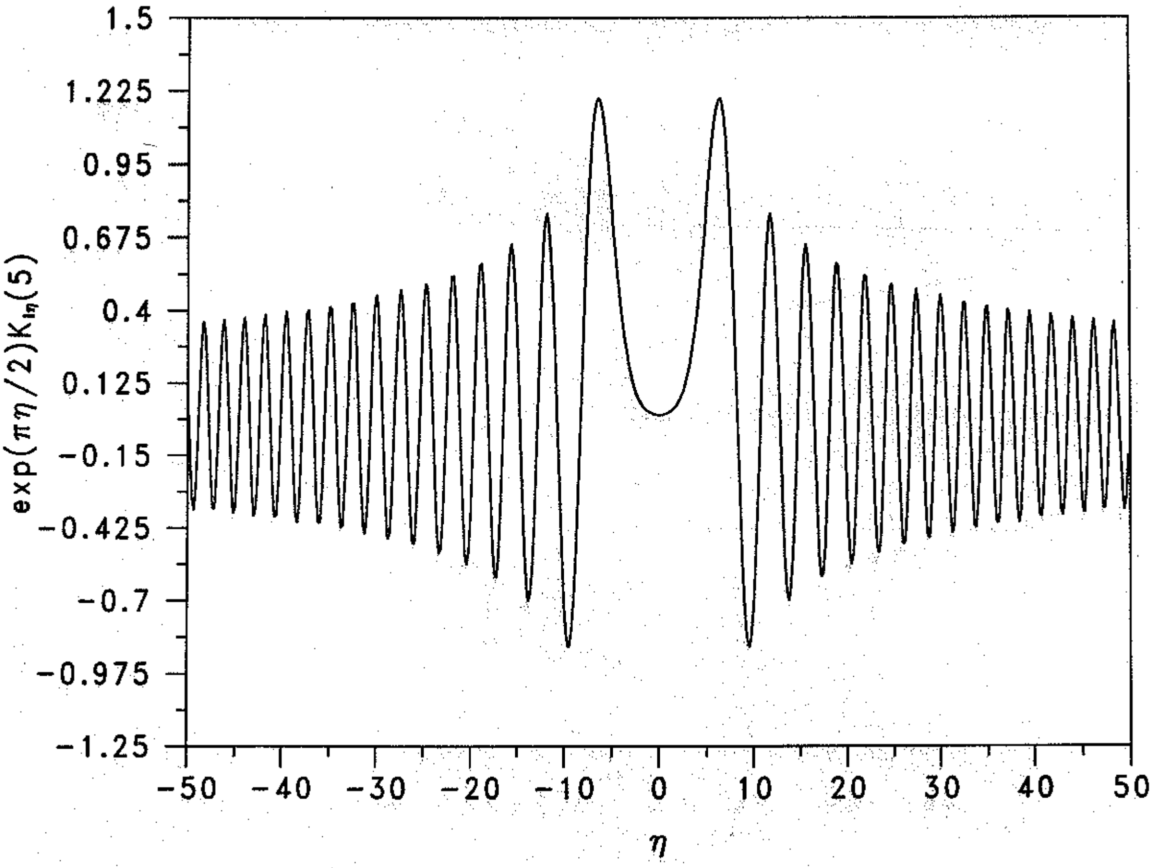

1.5
1.225
0.95
0.675
0.4
0.125
−0.15
−0.425
−0.7
−0.975
−1.25
exp(πη/2)K_iη(5)
−50
−40
−30
−20
−10
0
10
20
30
40
50
η

Fig. 2c

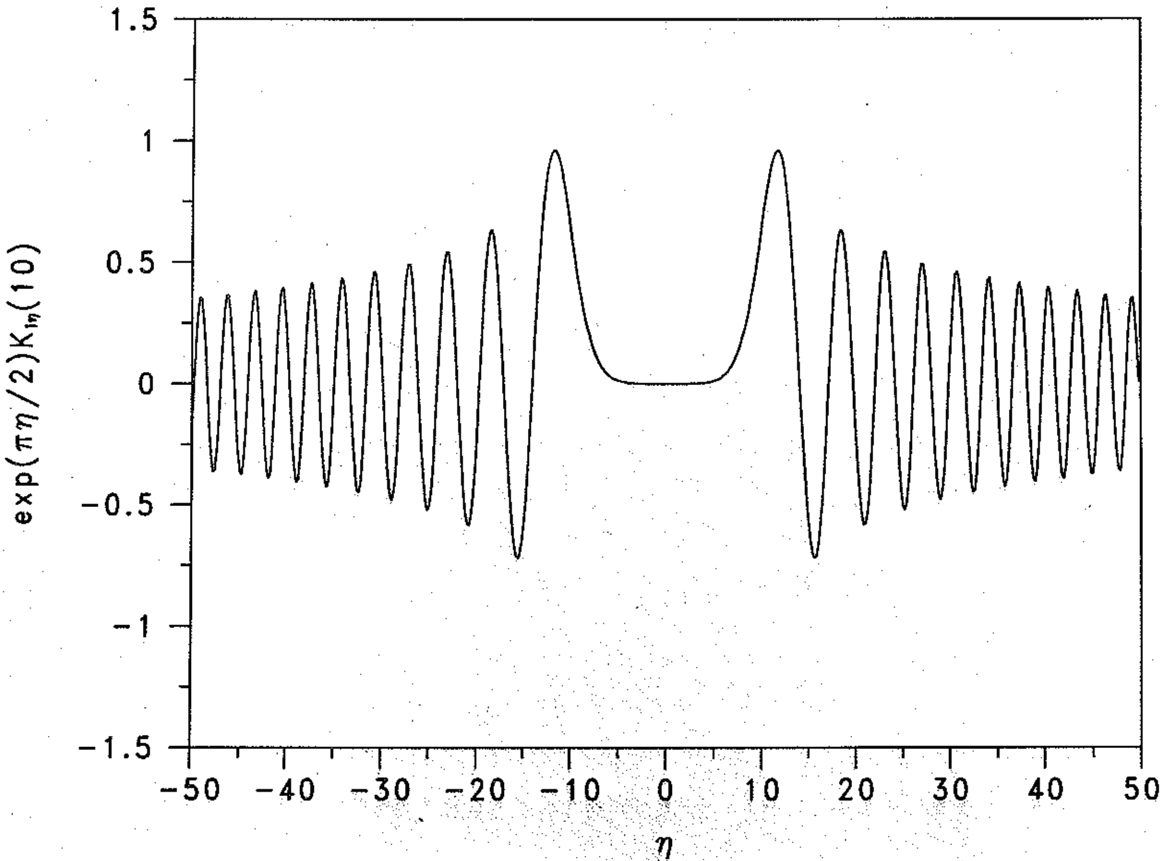
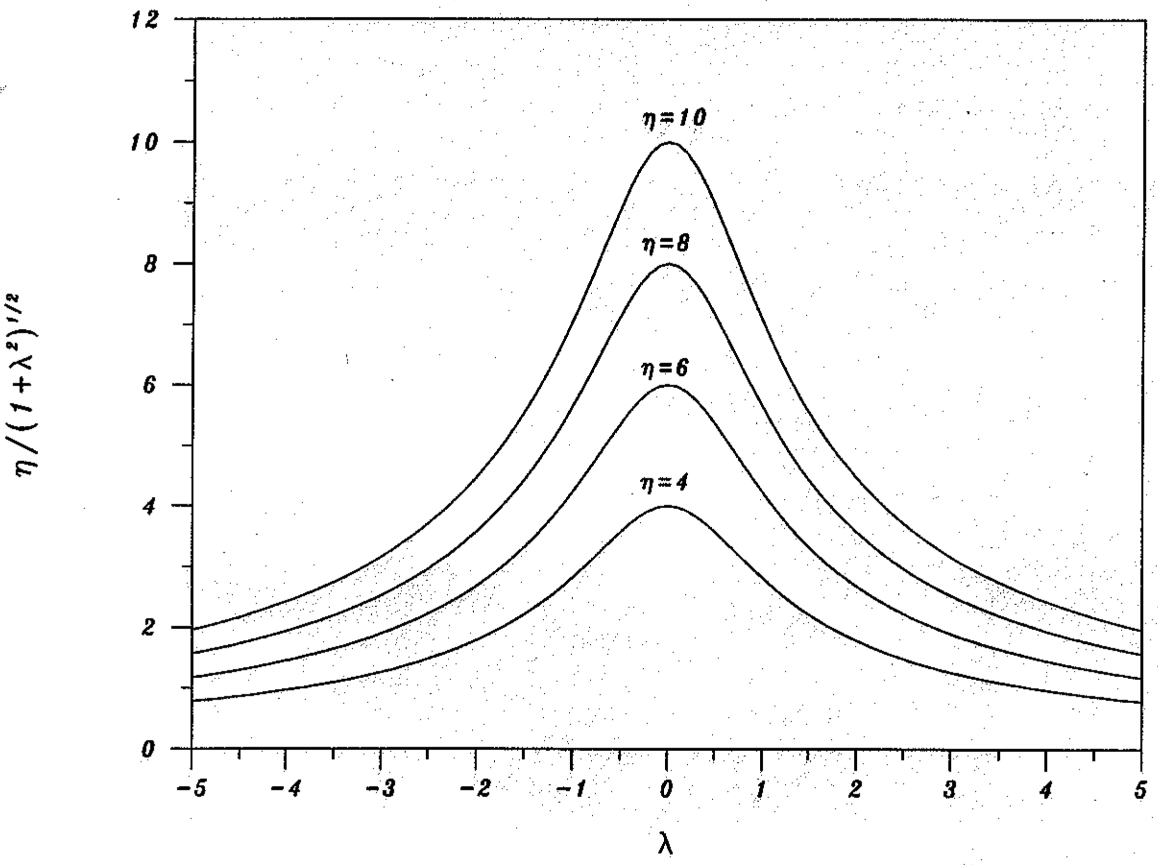
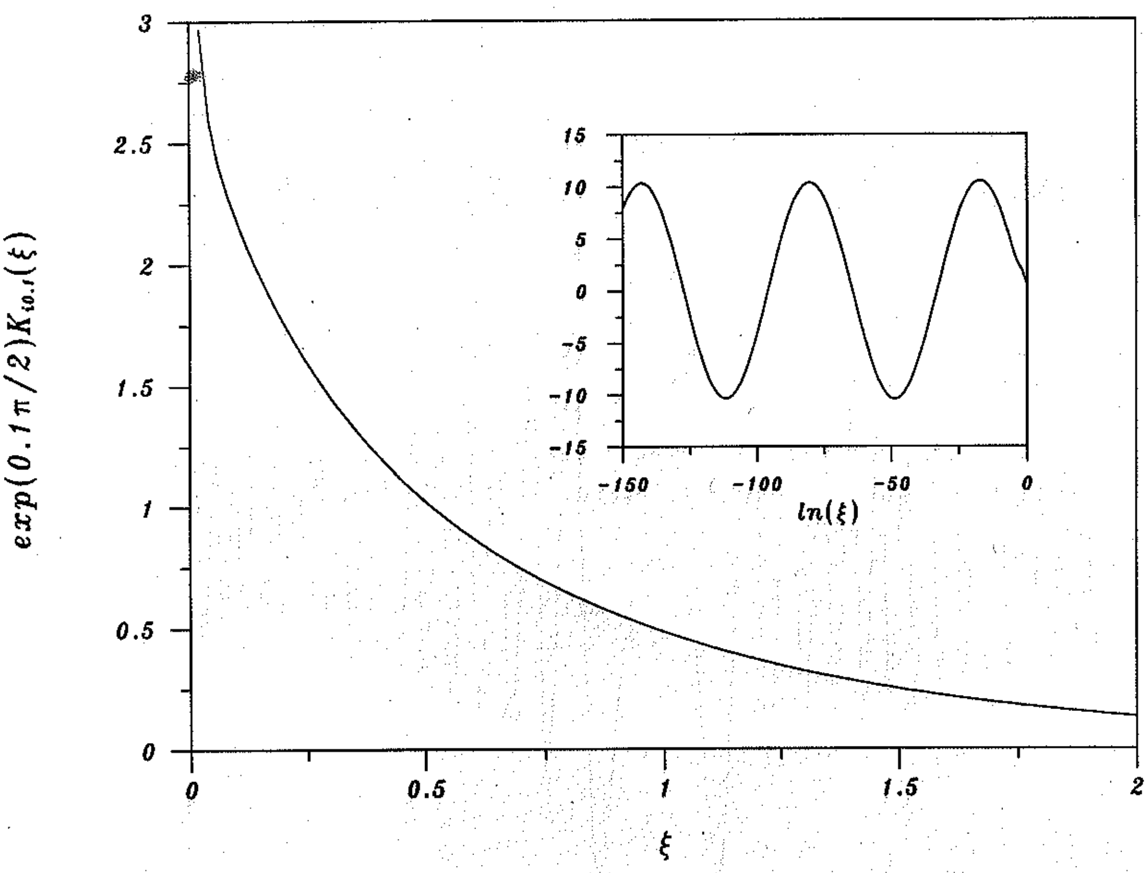

1.5
1
0.5
0
−0.5
−1
−1.5
exp(πη/2)K_iη(10)
−50
−40
−30
−20
−10
0
10
20
30
40
50
η

Fig. 2d



12
10
8
6
4
2
0
η=10
η=8
η=6
η=4
−5
−4
−3
−2
−1
0
1
2
3
4
5
λ
η/(1+λ²)^{1/2}

Fig. 3



3
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
ξ
exp(0.1π/2)K_{i0.1}(ξ)
15
10
5
0
−5
−10
−15
−150
−100
−50
0
ln(ξ)

Fig. 4a

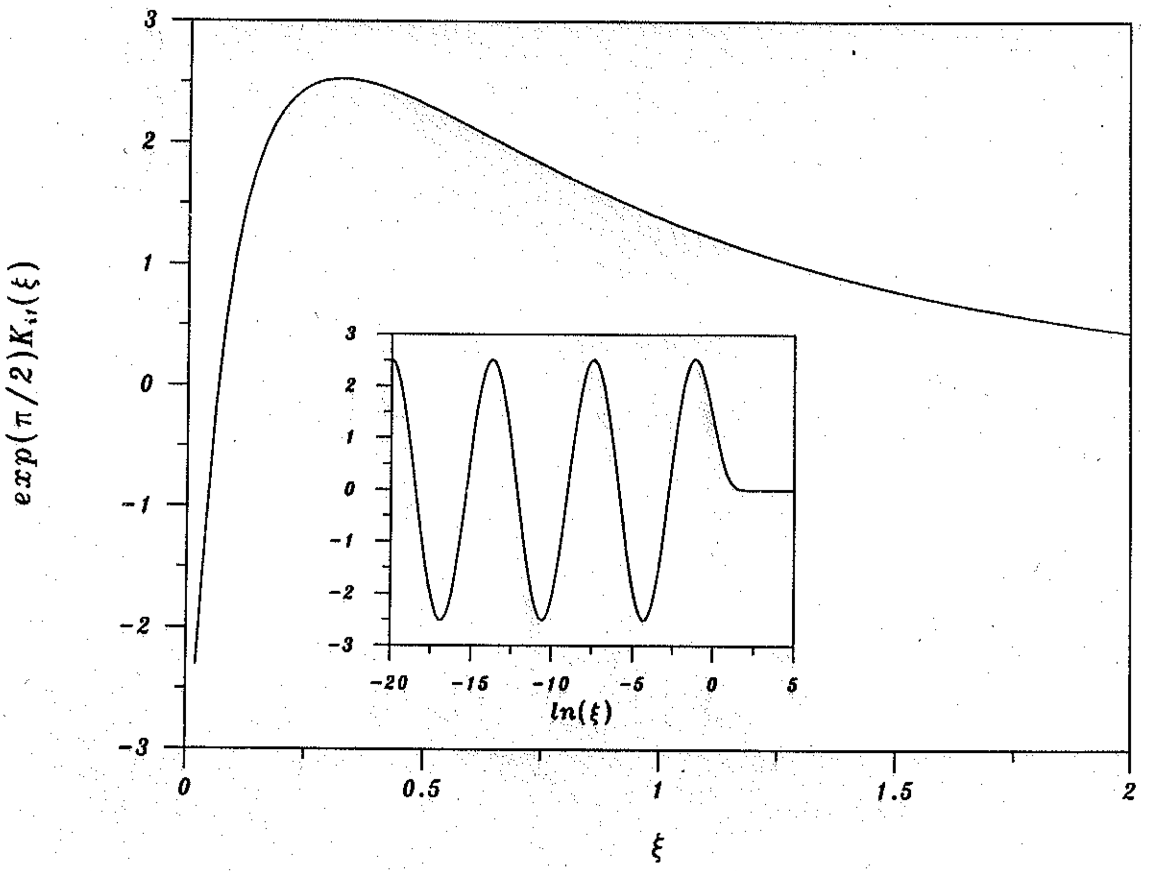


Fig. 4b

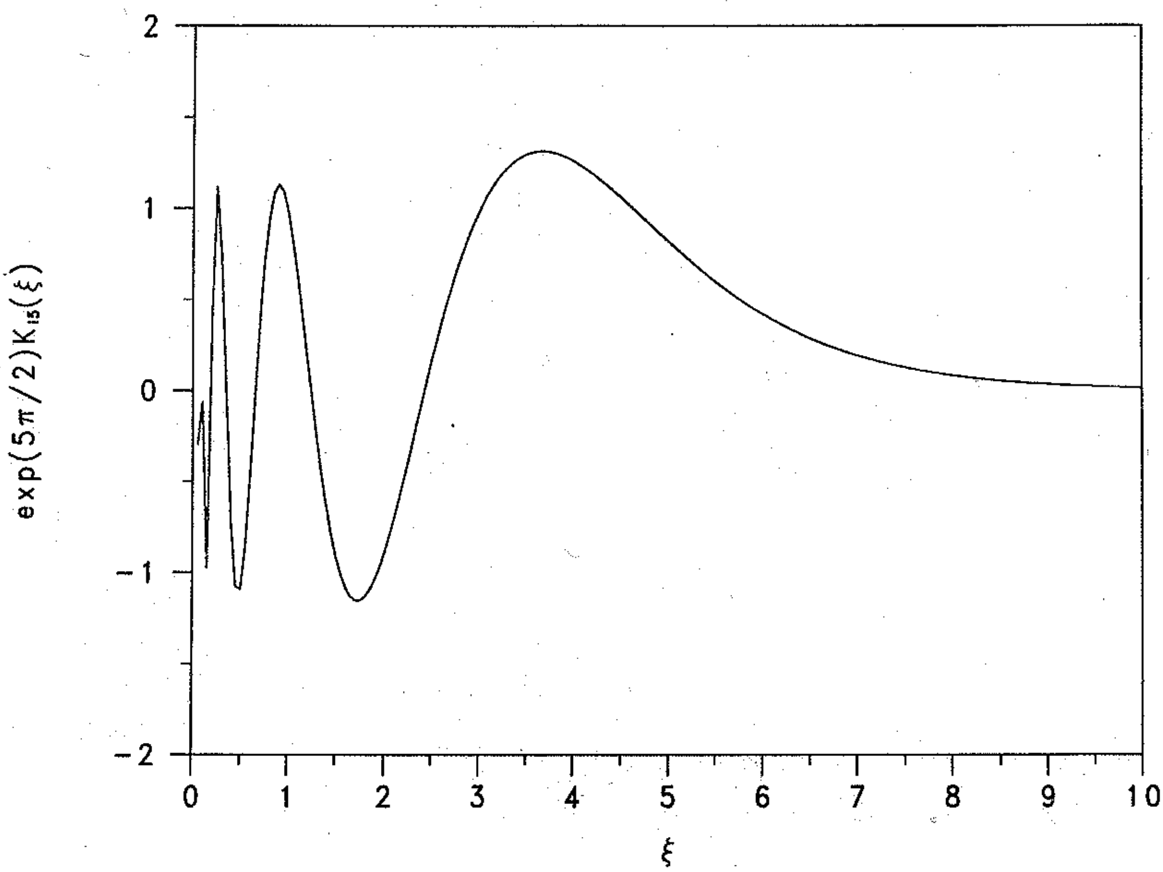


Fig. 4c

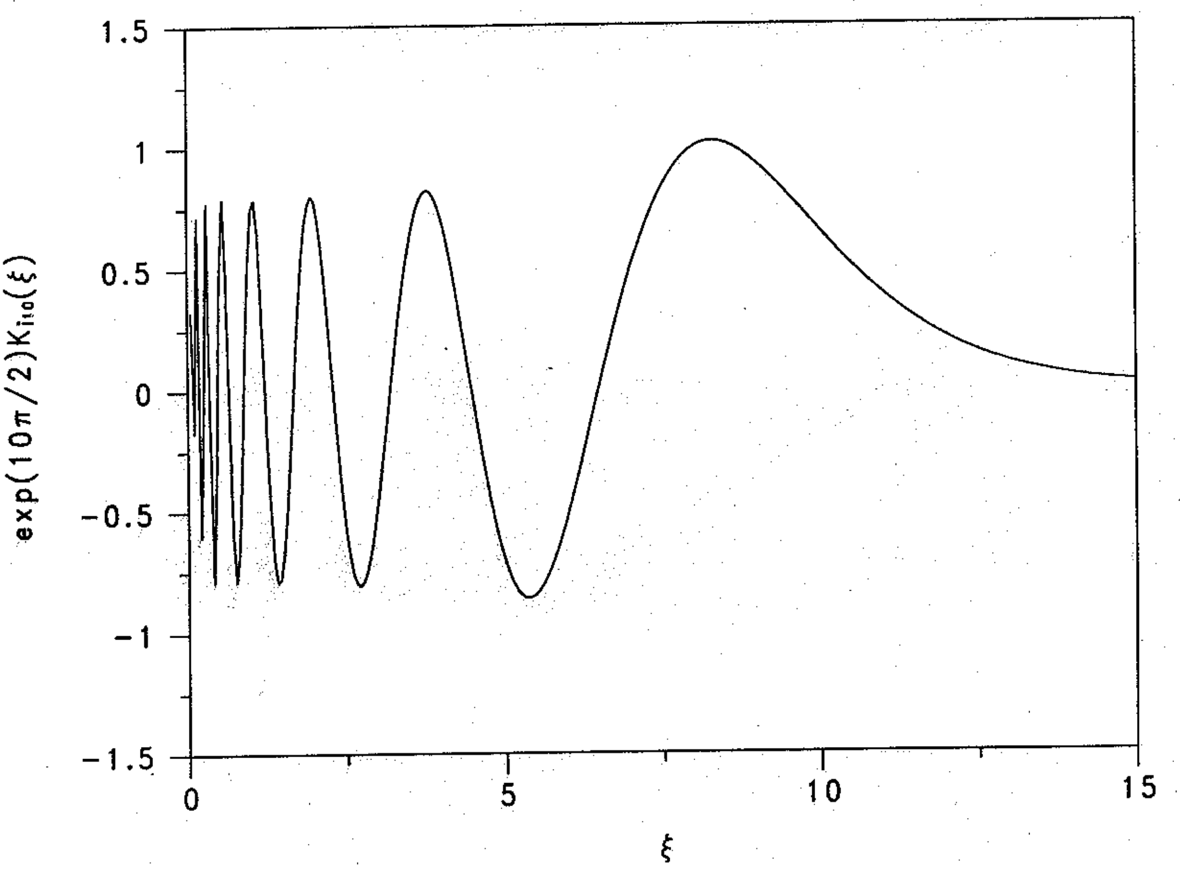


Fig. 4d