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SPECTRA OF HELICAL MODES PRODUCED BY TOROIDAL HELICAL CURRENTS

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ABSTRACT

An expression is derived for the magnetic scalar potential due to helical currents in toroidal surfaces. Nonlinearity of helical windings and noncircularity of the cross section are taken into account as the work resulted during an attempt to quickly evaluate the spectra of sinusoidal modes produced by instability currents in tokamaks. Taking the inverse aspect ratio as the order of magnitude reference parameter, a successive approximations method is developed that provides a solution with any desired accuracy.

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1. Introduction

The knowledge of the magnetic fields generated by specific toroidal helical currents is of interest in plasma physics of toroidal devices such as torsatrons, stellarators and tokamaks.

In stellarators and torsatrons the magnetic field configuration is determined by a number of current carrying conductors, wound on the chambers.

Some experiments on instabilities in tokamaks are performed with external helical currents (Pulsator team (1985)).

Experimental investigations have demonstrated that the m=2, n=1 tearing mode plays a crucial role in disruptures in tokamaks (Fussmann et al. (1980)). Currents flowing along field lines on a rational magnetic surface of a tokamak plasma in equilibrium state may form magnetic islands structure at each rational surface (Kucinski et al. (1991)) causing the disruption of the plasma. The formation of chains of islands in different rational surfaces is determined by the intensity of different helical modes; the position and the shape of the rational surfaces and the direction of the field lines are determined by the equilibrium conditions (solution of Grad–Shafranov equation). Therefore, it is interesting to have analytical expressions for the magnetic field due to some more general windings as found inside tokamak plasmas.

Basically three approaches have been taken up till now in similar calculations: direct determination of the magnetic field using the Biot–Savart law (W.N.–C. Sy (1981)), direct determination of the vector potential (Mirin et al. (1976)) or application of boundary conditions on the scalar potential expression (e.g. Kucinski & Caldas (1987)). In all the referred works the surface current is taken on a circular toroidal surface. The expressions for the vector quantities look awesome even in these simplest cases.

In order to determine the field due to nonlinear windings on noncircular toroids we made option for using the scalar potential approach.

In part 2 the expression for the surface current due to filamentary currents is written in terms of suitable curvilinear coordinates.

Boundary conditions for the scalar potential is derived in part 3.

In part 4 a method for quick evaluation of the spectra of helical modes is presented. By successive approximations the solution can be improved to a desired accurancy.

2. Surface current

Two systems of coordinates have been used: standard toroidal coordinates, in order to write the scalar potential as a solution of Laplace's equation and a non–orthogonal system where a set of the coordinate surfaces is formed by not necessarily circular toroids and the helical windings are coordinate curves.

2.1. The standard toroidal coordinates (ξ, ω, φ)

These are defined in terms of circular cylindrical coordinates by

$$R = \frac{R_0 \sinh \xi}{\cosh \xi - \cos \omega} \; ; \qquad Z = \frac{R_0 \sin \omega}{\cosh \xi - \cos \omega} \tag{1}$$

where R_0 is the major radius of the circular centre; φ is the toroidal angle. Equivalently,

$$\eta \equiv \cosh \xi \qquad \text{and} \qquad \theta_t \equiv \pi - \omega \tag{2}$$

are used because the scalar potential appears as a direct function of cosh ξ and because θ_t has the meaning of poloidal angle and is taken in the same direction as the local polar angle whereas ω is measured in opposite direction.

2.2. The helical windings

Explicit expressions for the boundary conditions on the scalar potential are most easily found if curvilinear coordinates x^i are used, where x^1 is a toroidal surface label and x^2=constant represents a modulated winding law.

We are especially concerned with currents flowing along magnetic fields in rational surfaces of a tokamak plasma in equilibrium. x^1 could be a magnetic surface label.

Here we assume noncircular toroidal surfaces described by:

$$\eta = \bar{\eta}(x^1) + \tilde{\eta}(x^1) \cos \theta_t \ . \tag{3}$$

For an axially symmetric equilibrium field $\vec{B}_0$ a local safety factor can be defined as (Kucinski et al. (1990)):

$$q \equiv \frac{d\varphi}{d\theta_t} = \frac{\vec{B}_0 \cdot \nabla\varphi}{\vec{B}_0 \cdot \nabla\theta_t} \quad . \tag{4}$$

In tokamaks this is very nearly:

$$q \simeq \bar{q}(x^1) + \tilde{q}(x^1) \cos \theta_t \quad . \tag{4'}$$

Filamentary currents are considered here on a rational surface along a line:

$$\frac{d\varphi}{d\theta_t} = \frac{m}{n} + \tilde{q}(x_0^1) \cos \theta_t \quad .$$

This modulated winding law is written in terms of the coordinate

$$x^2 \equiv m\,\theta_t - n\varphi + n\tilde{q} \sin \theta_t \tag{5}$$

as $x^2 = x_0^2 =$ constant.

We take $x^3 \equiv \theta_t$.

2.3. The current density

Filamentary currents on $x^1 = x_0^1$ surface along $x^2 = x_0^2$ can be described by a current density vector:

$$\vec{J} = C\,\delta(x^1 - x_0^1)\,\delta(x^2 - x_0^2)\,e_3 \quad . \tag{6}$$

An effort is made in order to use the most familiar notations in the litterature: e_i and e^i stand for covariant and contravariant bases vectors; g_{ij} and g^{ij} for the metric elements and g for the covariant determinant.

The flux through a surface element $d\vec{\sigma} = \sqrt{g}\,dx^1\,dx^2\,e^3$ must be the total current I_H . Thus:

$$\int \vec{J} \cdot d\vec{\sigma} = \int \sqrt{g}\,dx^1\,dx^2\,C\,\delta(x^1 - x_0^1)\,\delta(x^2 - x_0^2)$$

can be used in order to determine C as $I_H = \sqrt{g}\,C$.

The expression (6) becomes:

$$\vec{J} = \frac{I_H}{\sqrt{g}}\,\delta(x^1 - x_0^1)\,\delta(x^2 - x_0^2)\,e_3 \quad . \tag{6'}$$

If a single filament is considered at $x^2{=}0$, $\delta(x^2{-}x_0^2)$ can be written in a periodic form as:

$$\delta(x^2) = \frac{1}{2\pi} + \frac{1}{\pi}\,\mathrm{Re} \sum_{N=1}^{\infty} e^{iNx^2}$$

Re stands for real part. In what follows complex expressions are used. The physical quantities are obtained taking the real part.

If pairs of equally spaced conductors either in the same direction or in opposite directions are considered, the current density is still given as a series of terms in the form (Kucinski & Caldas (1987)):

$$\vec{J} = \frac{I_H}{\sqrt{g}}\,\delta(x^1 - x_0^1)\,\frac{e^{iNx^2}}{\pi} \quad . \tag{7}$$

Scalar potential for this current is determined here.

3. The boundary conditions

The boundary conditions on the magnetic field $\vec{B}$ can be written in a single expression:

$$[B] = -\mu_0 \int_i^e d\vec{r}_n \times \vec{J} \quad . \tag{8}$$

Square brackets are used to denote the jump in the enclosed quantity from inner to outer region, through the discontinuity surface. The suffixes e and i are used to designate the quantities in the outer and the inner regions, respectively.

$d\vec{r}_n$ is a displacement normal to x_0^1 surface and can be written:

$$d\vec{r}_n = \frac{e^1\,dx^1}{g^{11}} \quad . \tag{9}$$

The integral in (8) is performed through the discontinuity surface.

Using (7) and (9) the condition (8) becomes:

$$[\vec{B}] = \frac{\mu_0\, I_H}{\pi}\, e^{iNx^2}\left(e^2 - \frac{g^{12}}{g^{11}}\,e^1\right) \quad . \tag{10}$$

In terms of the scalar potential ϕ, $\vec{B} = \nabla\phi = \frac{\partial\phi}{\partial x^i}\,e^i$ and the boundary conditions are written:

$$\left[\frac{\partial\phi}{\partial x^1}\right] = -\frac{\mu_0\, I_H}{\pi}\, e^{iNx^2}\,\frac{g^{12}}{g^{11}}$$

$$\left[\frac{\partial\phi}{\partial x^2}\right] = \frac{\mu_0\, I_H}{\pi}\, e^{iNx^2} \tag{11}$$

$$\left[\frac{\partial\phi}{\partial x^3}\right] = 0 \quad .$$

Only two of these conditions are independent.

4. The scalar potential

Laplace's equation is separable in standard toroidal coordinates and the scalar potential is a superposition of terms:

$$\phi_\nu = (\eta + \cos\theta_t)^{1/2}\, Z_\nu(\eta)\, e^{i\nu\theta_t}\, e^{iN(m\theta_t - n\varphi)} \tag{12}$$

where $Z_\nu(\eta)$ are associated Legendre functions. Requiring that the potential be regular, inside the toroidal surface $Z_\nu^i \equiv Q_{mN+\nu-1/2}^{nN}(\eta)$ and in the outer region, $Z_\nu^e \equiv P_{mN+\nu-1/2}^{nN}(\eta)$.

If the winding law is given by (5) and $\frac{n}{m}\bar{q}$ and $\bar{n}/\bar{\eta}$ are of the order of the inverse aspect ratio ε, the contribution to the potential by each term φ_ν is of the order of $\varepsilon^{|\nu|}$. This is confirmed by the results.

The coordinate x^2 (5) is used to write the general expression for the potential as:

$$\phi = \frac{\mu_0 \, I_H}{iN\pi} \bar{\eta}_0^{-1/2} (\eta + \cos\theta_t)^{1/2} \sum_{\nu=-\infty}^{+\infty} \varepsilon^{|\nu|} C_\nu Z_\nu e^{i(\nu\theta_t - Nn\bar{q}\sin\theta_t)} e^{iNx^2} \tag{12'}$$

$\bar{\eta}_0$ is the value of $\bar{\eta}$ at the boundary.

Using $x^3 \equiv \theta_t$ and $\eta = \eta(x^1, \theta_t)$ the boudary conditions (11) are written:

$$\sum_{\nu=-\infty}^{+\infty} \left[\varepsilon^{|\nu|} C_\nu Z_\nu(\eta)\right] e^{i\nu\theta_t} = \bar{\eta}^{1/2} (\eta + \cos\theta_t)^{-1/2} e^{i\,Nn\bar{q}\sin\theta_t} \tag{13}$$

and

$$\sum_{\nu=-\infty}^{+\infty} \left[\varepsilon^{|\nu|} C_\nu Z'_\nu(\eta)\right] e^{i\nu\theta_t} = -\bar{\eta}^{1/2} \tfrac{1}{2} (\eta + \cos\theta_t)^{-3/2} e^{i\,Nn\bar{q}\sin\theta_t} +$$

$$-\bar{\eta}^{1/2}\, iN \frac{g^{12}}{g^{11}} \left(\frac{\partial\eta}{\partial x^1}\right)^{-1} (\eta + \cos\theta_t)^{-1/2} e^{i\,Nn\bar{q}\sin\theta_t} . \tag{13'}$$

The prime denotes derivative with respect to the argument.

Writting explicitly the surface equation (3) the second members are exactly expressed as Fourier series (Appendix A):

$$\sum_{\nu=-\infty}^{+\infty} \left[\varepsilon^{|\nu|} C_\nu Z_\nu(\eta)\right] e^{i\nu\theta_t} = \sum_{S=-\infty}^{+\infty} \varepsilon^{|S|} A_S e^{iS\theta} \tag{14}$$

and

$$\sum_{\nu=-\infty}^{+\infty} \left[\varepsilon^{|\nu|} C_\nu \bar{\eta} Z'_\nu(\eta)\right] e^{i\nu\theta_t} = \sum_{S=-\infty}^{+\infty} \varepsilon^{|S|} B_S e^{iS\theta_t} \tag{14'}$$

4.1. Circular toroidal boundary ($\bar{\eta} = 0$)

In this case, (14) become:

$$[C_S Z_S] \equiv C_S^e Z_S^e - C_S^i Z_S^i = A_S \tag{15}$$

and

$$[C_S \; \bar{\eta} \; Z'_S] \equiv C_S^e \; \bar{\eta} \; Z_S^{e\prime} - C_S^i \; \bar{\eta} \; Z_S^{i\prime} = B_S \tag{15'}$$

The constants are comfortably determined as:

$$C_S^e = \left(A_S - B_S \frac{Q}{\bar{\eta}\, Q'}\right) \frac{PQ'}{PQ' - P'Q} \frac{1}{P}$$

$$C_S^i = \left(A_S - B_S \frac{P}{\bar{\eta}\, P'}\right) \frac{P'Q}{PQ' - P'Q} \frac{1}{Q} \tag{16}$$

where $Z^{e,i}$ were subtituted by P,Q :

$$P \equiv P^{nN}_{mN+S-1/2}(\bar\eta)$$

$$Q \equiv Q^{nN}_{mN+S-1/2}(\bar\eta) \quad .$$

All the quantities are taken on the surface.

C_S are real numbers and the potential takes the form:

$$\phi = \frac{\mu_0 \; I_H}{N\pi} \left(\frac{\eta + \cos \theta_t}{\bar\eta_0} \right)^{1/2} \sum_{\nu=-\infty}^{+\infty} \varepsilon^{|\nu|} \; C_\nu \, Z_\nu(\eta) \sin((Nm + \nu)\, \theta_t - Nn\varphi) \quad .$$

It is evident that $C^e_S P$ and $C^i_S Q$ are of the same order of magnitude as A_S and B_S.

The expression is more general than in above cited papers as it fully takes into account the nonlinearity of the winding law $(\bar q \neq 0)$. The intensity of each helical mode can be evaluated with any desired accuracy.

4.2. Elliptic toroidal boundary ($\bar\eta \neq 0$)

Legendre functions are expanded in Fourier series (Appendix B):

$$Z_\nu(\bar\eta + \tilde\eta \cos \theta_t) = Z_{\nu,0}(\bar\eta) + \sum_{\ell=1}^{\infty} \varepsilon^\ell Z_{\nu,\ell}\left(e^{i\ell\theta_t} + e^{-i\ell\theta_t}\right) \tag{17}$$

to derive:

$$\sum_{\nu=-\infty}^{+\infty} \left[\varepsilon^{|\nu|} \; C_\nu \, Z_\nu(\bar\eta + \tilde\eta \cos \theta_t)\right] e^{i\nu\theta_t} = \sum_{S=-\infty}^{+\infty} e^{iS\theta_t} \left\{ \varepsilon^{|S|} \left[C_S \, Z_{S,0}(\bar\eta)\right] + \right.$$

$$\left. + \sum_{\ell=1}^{\infty} \left[\varepsilon^{|S-\ell|+\ell} C_{S-\ell} Z_{S-\ell,\ell} + \varepsilon^{|S+\ell|+\ell} C_{S+\ell} Z_{S+\ell,\ell}\right]\right\} \quad . \tag{18}$$

Using analogous series for the derivatives. equations (14) yield:

$$\varepsilon^{|S|}\left[C_S \, Z_{S,0}\right] = \varepsilon^{|S|} A_S - \sum_{\ell=1}^{\infty} \varepsilon^{|S\mp\ell|+\ell}\left[C_{S\mp\ell} Z_{S\mp\ell,\ell}\right] \equiv \varepsilon^{|S|} \mathcal{A}_S$$

$$\varepsilon^{|S|}\left[C_S \, \bar\eta \, Z'_{S,0}\right] = \varepsilon^{|S|} B_S - \sum_{\ell=1}^{\infty} \varepsilon^{|S\mp\ell|+\ell}\left[C_{S\mp\ell} \, \bar\eta \, Z'_{S\mp\ell,\ell}\right] \equiv \varepsilon^{|S|} \mathcal{B}_S \quad . \tag{19}$$

The solutions can be formally written similar to (16):

$$C^e_S = \left(\mathcal{A}_S - \mathcal{B}_S \frac{\mathcal{Q}}{\bar\eta \, \mathcal{Q}'} \right) \frac{\mathcal{P}\mathcal{Q}'}{\mathcal{P}\,\mathcal{Q}' - \mathcal{P}'\mathcal{Q}} \, \frac{1}{\mathcal{P}}$$

$$C^i_S = \left(\mathcal{A}_S - \mathcal{B}_S \frac{\mathcal{P}}{\bar\eta \, \mathcal{P}'} \right) \frac{\mathcal{P}'\mathcal{Q}}{\mathcal{P}\,\mathcal{Q}' - \mathcal{P}'\mathcal{Q}} \, \frac{1}{\mathcal{Q}} \tag{20}$$

where $\mathcal{P} \equiv Z^e_{S,0} = P^{nN}_{mN+S-1/2}(\bar\eta) + O(\varepsilon^2)$

and $\mathcal{Q} \equiv Z^i_{S,0} = Q^{nN}_{mN+S-1/2}(\bar\eta) + O(\varepsilon^2)$.

First order approximation

Keeping the dominant terms in (19) $\mathcal{A}_S$ are derived as:

$$\mathcal{A}_0 = A_0 + O(\varepsilon^2)$$

$$\mathcal{A}_S = A_S - \sum_{\ell=1}^{S} \left[C_{S-\ell} Z_{S-\ell,\ell}\right] + O(\varepsilon^2) \quad \text{for} \quad S \geq 1 \qquad (21)$$

$$\mathcal{A}_S = A_S - \sum_{\ell=1}^{|S|} \left[C_{-|S|+\ell} Z_{-|S|+\ell,\ell}\right] + O(\varepsilon^2) \quad \text{for} \quad S \leq -1 \; .$$

Similar expressions are derived for $\mathcal{B}_S$.

The first order coefficients C_S are determined following the scheme:

$$\left(\mathcal{A}_0 \, , \mathcal{B}_0\right) \to C_0 \to \left(\mathcal{A}_{\pm 1} \, , \mathcal{B}_{\pm 1}\right) \to C_{\pm 1} \to \dots \to \left(\mathcal{A}_{\pm S} \, , \mathcal{B}_{\pm S}\right) \to C_{\pm S} \qquad (22)$$

The scalar potential with the first terms in this approximation is written in Appendix C.

Successive approximations method

Once the first order coefficients are evaluated $\left(\mathcal{A}_S \, . \, \mathcal{B}_S\right)$ are written in terms of these coefficents using (19) in order to determine the next order solution. This procedure can be repeated until the desired accuracy is achieved.

Conclusions

The magnetic structure and consequently the stability of toroidal plasmas are strongly dependent upon the helical modes of the magnetic field (La Haye et al. (1981), Kucinski et al. (1991)). Very often it is more relevant to have the approximate values of the whole spectrum of helical modes in the plasma region rather than a higly accurate value of the magnetic field especially if resonance phenomena are concerned (Kucinski et al. (1991)).

The present work provides a method for quick determination of these spectra. A general expression for the magnetic field has not been derived as the relevant components of the magnetic field differ from case to case. These can be obtained without additional complication switching to appropriate coördinates. In tokamaks the widths of the magnetic islands depend upon the component perperdicular to rational magnetic surfaces; equilibrium magnetic surface coordinates are most suited in this case.

Appendix A

The basic identities:

$$(\eta + \cos\theta_t)^{-1/2} = \frac{\sqrt{2}}{\pi} \sum_{\ell=-\infty}^{+\infty} (-1)^{\ell} Q_{|\ell|-1/2}(\eta)\, e^{i\ell\theta_t} \tag{A.1}$$

$$(\eta + \cos\theta_t)^{-3/2} = -2 \frac{\partial}{\partial\eta} (\eta + \cos\theta_t)^{-1/2} \tag{A.2}$$

$$e^{iN\tilde{q}\sin\theta_t} = \sum_{k=-\infty}^{+\infty} J_K(N\tilde{q})\, e^{iK\theta_t} \tag{A.3}$$

$$\frac{\sin\omega}{\cosh\xi - \cos\omega} = 2 \sum_{n=1}^{\infty} e^{-n\xi} \sin n\omega \tag{A.4}$$

are used here. $Q_{\ell-1/2}$ are associated Legendre functions and J_k are cylindrical Bessel functions.

A_S and B_S are defined by:

$$\bar{\eta}^{-1/2} \sum_{S=-\infty}^{+\infty} \varepsilon^{|S|} A_S\, e^{iS\theta_t} \equiv (\eta + \cos\theta_t)^{-1/2} e^{iN\tilde{q}\sin\theta_t} \tag{A.5}$$

$$\bar{\eta}^{-3/2} \sum_{S=-\infty}^{+\infty} \varepsilon^{|S|} B_S^1\, e^{iS\theta_t} \equiv -\frac{1}{2} (\eta + \cos\theta_t)^{-3/2} e^{iN\tilde{q}\sin\theta_t} \tag{A.6}$$

$$\bar{\eta}^{-3/2} \sum_{S=-\infty}^{+\infty} \varepsilon^{|S|} B_S^2\, e^{iS\theta_t} \equiv -iN \frac{g^{12}}{g^{11}} \left(\frac{\partial\eta}{\partial x^1}\right)^{-1} (\eta + \cos\theta_t)^{-1/2} e^{iN\tilde{q}\sin\theta_t} \tag{A.7}$$

and $B_S = B_S^1 + B_S^2$. As $\eta = \bar{\eta} + \tilde{\eta}\cos\theta_t$:

$$(\eta + \cos\theta_t)^{-1/2} = (\tilde{\eta} + 1)^{-1/2} \left(\frac{\bar{\eta}}{\tilde{\eta} + 1} + \cos\theta_t\right)^{-1/2}$$

and

$$-\frac{1}{2} (\eta + \cos\theta_t)^{-3/2} = -\frac{1}{2} (\tilde{\eta} + 1)^{-3/2} \left(\frac{\bar{\eta}}{\tilde{\eta} + 1} + \cos\theta_t\right)^{-3/2} .$$

Using (A.1) to (A.3) the expressions for A_S and B_S^1 are easily obtained as:

$$\varepsilon^{|S|} A_S = \frac{\sqrt{2}}{\pi} \sum_{\ell=-\infty}^{+\infty} (-1)^{\ell} \zeta^{1/2} Q_{|\ell|-1/2}(\zeta)\, J_{S-\ell}(N\tilde{q}) \tag{A.8}$$

and

$$\varepsilon^{|S|} B_S^1 = \frac{\sqrt{2}}{\pi} \sum_{\ell=-\infty}^{+\infty} (-1)^{\ell} \zeta^{3/2} Q'_{\ell-1/2}(\zeta)\, J_{S-\ell}(N\tilde{q}) \tag{A.9}$$

where

$$\zeta \equiv \frac{\bar{\eta}}{\tilde{\eta} + 1} .$$

In order to determine B_S^2, x^1 and x^2 are explicitly written in terms of standard toroidal coordinates (ξ, ω, φ):

$$\eta \equiv \cosh\xi = \bar{\eta}(x^1) + \tilde{\eta}(x^1) \cos\theta_t$$

$$x^2 = m\,\theta_t - n\varphi + n\tilde{q}\sin\theta_t$$

where $\theta_t = \pi - \omega$.

Using:

$$g^{11} = \nabla x^1 \cdot \nabla x^1 \quad ; \quad g^{12} = \nabla x^1 \cdot \nabla x^2$$

$$\nabla\xi\cdot\nabla\xi = \nabla\theta_t\cdot\nabla\theta_t \quad ; \quad \nabla\xi\cdot\nabla\theta_t = \nabla\varphi\cdot\nabla\theta_t = 0$$

and

$$\sinh\xi\,\nabla\xi = \frac{\partial\eta}{\partial x^1}\nabla x^1 - \bar{\eta}\sin\theta_t\,\nabla\theta_t$$

the following expression is derived:

$$\frac{g^{12}}{g^{11}}\left(\frac{\partial\eta}{\partial x^1}\right)^{-1} = \frac{1}{2\bar{\eta}}\,\frac{\sin\theta_t}{\cosh\alpha + \cos\theta_t}\,(m + n\tilde{q}\cos\theta_t) =$$

$$= \frac{m}{\bar{\eta}}\,e^{-\alpha}\left(1 - \frac{n}{m}\frac{\tilde{q}}{2}e^{-\alpha}\right)\sin\theta_t - \frac{m}{\bar{\eta}}\left(1 - \frac{n}{m}\tilde{q}\cosh\alpha\right)\sum_{\ell=2}^{\infty}(-1)^{\ell}e^{-\ell\alpha}\sin\ell\,\theta_t \tag{A.10}$$

where

$$\cosh\alpha \equiv \frac{\bar{\eta}^2 + \bar{\bar{\eta}}^2 - 1}{2\,\bar{\eta}\,\bar{\bar{\eta}}}\ .$$

Using (A.5), (A.8) and (A.10) in (A.7) and reordering the terms, an exact expression for B_S^2 is proved to be:

$$\varepsilon^{|S|}B_S^2 = \frac{mN}{2}e^{-\alpha}\left(-1 + \frac{n}{m}\frac{\tilde{q}}{2}e^{-\alpha}\right)\left(\varepsilon^{|S-1|}A_{S-1} - \varepsilon^{|S+1|}A_{S+1}\right) +$$

$$+ \frac{mN}{2}\left(1 - \frac{n}{m}\tilde{q}\cosh\alpha\right)\sum_{\ell=2}^{\infty}(-1)^{\ell}e^{-\ell\alpha}\left(\varepsilon^{|S-\ell|}A_{S-\ell} - \varepsilon^{|S+\ell|}A_{S+\ell}\right)\ .$$

Mathematical table edited by Erdelyi (1953) was most often consulted for formulae.

Appendix B

$$Z_\nu(\bar\eta + \tilde\eta \cos\theta_t) = \sum_{m=0}^{\infty} \frac{Z_\nu^{(m)}}{m!} (\tilde\eta \cos\theta_t)^m \qquad \text{(B.1)}$$

where

$$Z_\nu^m \equiv \frac{d^m}{d\bar\eta^m} Z_\nu(\bar\eta) .$$

The identity:

$$(\cos\theta_t)^m = 2^{-m} (e^{i\theta_t} + e^{-i\theta_t})^m = 2^{-m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} e^{i(2k-m)\theta_t}$$

is used in (B.1) and the terms are reordered to give (17) with:

$$\varepsilon^\ell Z_{\nu,\ell} = \sum_{k=0}^{\infty} \frac{Z_\nu^{(2k+\ell)}}{k!(k+\ell)!} \left(\frac{\tilde\eta}{2}\right)^{2k+\ell} , \quad \ell \geq 0 . \qquad \text{(B.2)}$$

Appendix C

Potential in first order approximation.

$$\phi \simeq \frac{\mu_0 I_H}{N\pi} \left(\frac{\eta + \cos\theta_t}{\bar\eta_0}\right)^{1/2} \sum_{\nu=-\infty}^{+\infty} M_\nu \frac{Z_\nu(\eta)}{Z_\nu(\bar\eta_0)} \sin\big((mN+\nu)\theta_t - Nn\varphi\big)$$

$$M_0^e = \frac{J_0}{2} ; \quad M_0^i = -\frac{J_0}{2}$$

$$M_1^e = \frac{J_1}{2} - \frac{mN}{4} \frac{\tilde\eta}{\bar\eta} J_0 - \frac{mN}{mN+1} \frac{1}{8\bar\eta} J_0$$

$$M_{-1}^e = -\frac{J_1}{2} - \frac{mN}{4} \frac{\tilde\eta}{\bar\eta} J_0 - \frac{mN-2}{mN-1} \frac{1}{8\bar\eta} J_0$$

$$M_1^i = -\frac{J_1}{2} - \frac{mN}{4} \frac{\tilde\eta}{\bar\eta} J_0 + \frac{mN+2}{mN+1} \frac{1}{8\bar\eta} J_0$$

$$M_{-1}^i = \frac{J_1}{2} - \frac{mN}{4} \frac{\tilde\eta}{\bar\eta} J_0 + \frac{mN}{mN-1} \frac{1}{8\bar\eta} J_0 .$$

All the quantities M_ν are evaluated on the discontinuity surface.

$J_k \equiv J_k(Nn\tilde q)$ are Bessel functions.

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