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KINETIC APPROACH TO THE INITIAL VALUE PROBLEM IN ϕ^4 FIELD THEORY

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ABSTRACT

A time-dependent projection technique is used to develop kinetic equations in the context of ϕ^4 field theory. A mean-field expansion can be written for these equations which are numerically tractable in the few lower orders. The procedure is applied to the case of the spatially uniform system in $1+1$ dimensions, including numerical solutions.

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1. Introduction

Interest on the initial value problem for quantum field theoretical models over the last decade stems mainly from two different areas of physics: on the one hand, the inflationary scenario of the early universe involves the control of the time-evolution of a driving scalar field[1]; on the other hand, properties of hadronic matter manifest themselves through transient phenomena in globally off-equilibrium situations in high-energy collisions[2]. In both cases non-perturbative methods must be employed.

A natural starting point in either of the two contexts has been provided by self-consistent kinetic descriptions based on a mean-field approximation. In fact, both the variational method using Gaussian trial functionals[3] and the so called Hartree and Hartree-Fock approximations[4] fall into this category. Corrections to these collisionless kinetic schemes, on the other hand, have been considered repeatedly in the context of nuclear dynamics [5]. In this work we follow an approach developed earlier in that context[6] to study collisional corrections to the kinetics of a self-interacting quantum-field. The approach is based on time-dependent projection techniques and leads to a mean-field expansion which reproduces the results obtained in the Gaussian functional approximation in its lowest order[7]. We are able to explicitly include and evaluate dynamical correlation effects which manifest themselves through suitable memory collision integrals added to the kinetic equations. The main effect of these corrections to the bare mean-field picture is to produce changes in time of the coherence properties of the initial state which in turn allow for a qualitative improvement in the description of the time evolution of some relevant dynamical variables. This point is illustrated by the discussion of numerical solutions both in $0+1$ and in $1+1$ dimensions in section VI below.

In section II we set up the general kinetic scheme adopted here for the case of a real self-interacting scalar quantum field. The projection technique and the related approximations for the dynamical correlation effects are described in section III. Section IV deals with calculations within the lowest, mean-field, approximations, and in particular with the adopted renormalization scheme. The full collisional approximation is explicitly given in section V for the case of spatially homogeneous systems. Section VI shows the numerical solutions of equations of motion obtained in section V. Some points of a more technical nature concerning the construction of projection operators and the numerical work are given in the Appendices.

II. Kinetics of a Self-Interacting Quantum Field

In this section, we shall describe a formal treatment of the kinetics of a self-interacting quantum field. Although the procedure is quite general, we will adopt the simplest context of a single scalar field in $1+1$ dimensions and assume spatial uniformity. This will illustrate all the relevant points of the approach and cut down inessential technical complications. Features of more general contexts are discussed in ref. [8] and briefly outlined in section VI.

The general idea of our approach[6] is to focus on the time development of a restricted set of simple observables. On the basis of the general dynamics of the field, we then derive an effective dynamics for them, which will be eventually expressed in terms of formal equations of kinetic type. A systematic expansion scheme can then be devised for these equations[9] which yields numerically tractable approximations of various orders. The lowest of these approximations, in particular, is equivalent to the Gaussian approximation currently used in connection with the variational formulation of the functional Schrödinger approach[7].

In order to implement this, the Heisenberg field operator $\phi(x)$ and the canonical momentum $\pi(x)$ are first Fourier expanded as

$$\begin{aligned} \phi(x) &= \sum_k [v_k(x)\,\gamma_k(t) + v_k^*(x)\,\gamma_k^+(t)] \quad , \\ \pi(x) &= -i\sum_k k_0[v_k(x)\,\gamma_k(t) - v_k^*(x)\,\gamma_k^+(t)] \quad , \end{aligned} \tag{2.1}$$

so that γ_k, γ_k^+ are annihilation and creation parts satisfying boson commutation

relations at equal times

$$[\gamma_k(t)\,,\,\gamma_{k'}^+(t)] \;=\; \delta_{kk'} \quad . \tag{2.2}$$

The $v_k(x)$ are the periodic boundary condition plane waves

$$v_k(x) \;=\; \frac{e^{ik\cdot x}}{\sqrt{2Lk_0}} \quad , \tag{2.3}$$

L being the lenght of the periodicity box. Here x is the spatial coordinate only and

$$k_0^2 \;=\; k^2 + \mu^2 \quad , \tag{2.4}$$

μ being an expansion mass parameter to be fixed latter in a convenient way.

The state of the system (assumed uniform) is described in terms of a density matrix F in the Heisenberg picture.

The first variable of interest is the expectation value of the field operator, $\mathrm{Tr}\,\phi(x)\,F$. In terms of the expansion (2.1) this is related to the quantities

$$\Gamma_k(t) \;=\; \mathrm{Tr}\,\gamma_k\,F \quad , \tag{2.5}$$

which can be interpreted as amplitudes of coherent condensates. In a spatially uniform system only $\Gamma_0(t)$ is different from zero. With the help of the Γ_k we can now define the shifted boson operators with vanishing expectation value in F

$$\beta_k(t) \;=\; \gamma_k(t) - \Gamma_k(t) \tag{2.6}$$

and include as variables of interest also the expectation values of pairs of β, β^+ operators at equal times, namely

$$R_{kk'}(t) \;=\; \mathrm{Tr}\,\beta_{k'}^+(t)\,\beta_k(t)\,F \quad \xrightarrow{\text{uniform system}} \quad p_k(t)\,\delta_{kk'} \tag{2.7a}$$

$$\Pi_{kk'}(t) \;=\; \mathrm{Tr}\,\beta_{k'}(t)\,\beta_k(t)\,F \quad \xrightarrow{\text{uniform system}} \quad r_k(t)\,\delta_{k,-k'} \quad . \tag{2.7b}$$

The hermitean matrix $\mathbf{R}$ and the symmetric matrix $\mathbf{\Pi}$ are in fact the one-boson density matrix and the pairing density for the shifted bosons. The corresponding matrices for the γ-bosons are of course easily expressed in terms of $\mathbf{R}$, $\mathbf{\Pi}$ and of the $\Gamma_k(t)$.

An important point is that the plane-waves (2.3) are natural orbitals of the one-boson density $\mathbf{R}$, which is all one has to deal with when the freedom associated with the pairing density is not included. In order to handle the pairing density one sets up, as usual[10], an extended density

$$\mathcal{R}_k(t) = \begin{pmatrix} R_k(t) & \Pi_k(t) \\ \Pi^*_{-k}(t) & 1+R_{-k}(t) \end{pmatrix} \quad , \quad \text{(uniform system)} \tag{2.8}$$

from which one obtains extended natural orbitals incorporating information on the pair density by solving the eigenvalue problem

$$\mathbf{G}\,\mathcal{R}_k\,\mathbf{X}_k = \mathbf{X}_k\,\mathbf{G}\,\mathbf{N}_k \tag{2.9}$$

where

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{X}_k = \begin{pmatrix} x_k & y_k^* \\ y_k & x_k^* \end{pmatrix} \quad \mathbf{N}_k = \begin{pmatrix} \nu_k & 0 \\ 0 & 1+\nu_k \end{pmatrix} \quad . \tag{2.10}$$

The eigenvalues ν_k can be interpreted as shifted boson ocupation numbers for the paired natural orbitals described by $\mathbf{X}_k$. From reflection symmetry one must have $\nu_k = \nu_{-k}$.

Since (2.9) is a non-hermitean eigenvalue problem, it is useful to consider also the adjoint equation

$$\mathcal{R}_k\,\mathbf{G}\,\tilde{\mathbf{X}}_k = \tilde{\mathbf{X}}_k\,\mathbf{G}\,\mathbf{N}_k \tag{2.11}$$

from which one finds that

$$\tilde{\mathbf{X}}_k = \mathbf{G}\,\mathbf{X}_k \quad . \tag{2.12}$$

The adjoint vectors $\tilde{\mathbf{X}}_k$ satisfy biorthogonality relations with the $\mathbf{X}_k$ which allow one to introduce the normalization condition

$$\tilde{\mathbf{X}}_k^+\,\mathbf{X}_k = \mathbf{X}_k^+\,\mathbf{G}\,\mathbf{X}_k = \mathbf{G} \tag{2.13}$$

and the completeness relation

$$\mathbf{X}_k\,\mathbf{G}\,\mathbf{X}_k^+ = \mathbf{G} \quad . \tag{2.14}$$

Furthermore, one can use the paired natural orbitals to construct new boson operators η_k, η_k^+ and shift amplitudes A_k, A_k^* as

$$\begin{pmatrix} \eta_k \\ \eta^+_{-k} \end{pmatrix} = \mathbf{X}_k^+ \begin{pmatrix} \beta_k \\ \beta^+_{-k} \end{pmatrix} \quad ; \quad \begin{pmatrix} A_k \\ A^*_{-k} \end{pmatrix} = \mathbf{X}_k^\dagger \begin{pmatrix} \Gamma_k \\ \Gamma^*_{-k} \end{pmatrix} \tag{2.15}$$

which can be inverted with the help of equation (2.14).

The next step is study the time-derivatives of the various relevant variables in terms of the Heisenberg equation of motion for $\phi(x)$. For the $\Gamma_k(t)$ one finds imediately

$$i\dot{\Gamma}_k = \mathrm{Tr}[\gamma_k, H]\,F = x_k\,\mathrm{Tr}[\eta_k, H]\,F - y_k^*\,\mathrm{Tr}[\eta^*_{-k}, H]\,F \quad , \tag{2.16}$$

H being the field Hamiltonian. As for the remaining quantities, we first rewrite the eigenvalue equation (2.9), using eq. (2.13), as

$$\mathbf{X}_k^+\,\mathcal{R}_k\,\mathbf{X}_k = \mathbf{N}_k \tag{2.17}$$

from which it follows that

$$\mathbf{X}_k^+\,\dot{\mathcal{R}}_k\,\mathbf{X}_k = \dot{\mathbf{N}}_k - \dot{\mathbf{X}}_k^+\,\mathcal{R}_k\,\mathbf{X}_k - \mathbf{X}_k^+\,\mathcal{R}_k\,\dot{\mathbf{X}}_k \quad , \tag{2.18}$$

we now evaluate the left hand side of this equation using the Heisenberg equation of motion to obtain

$$i\,\mathbf{X}_k^+\,\dot{\mathcal{R}}_k\,\mathbf{X}_k = \begin{pmatrix} \mathrm{Tr}[\eta_k^+\,\eta_k\,, H]\,F & \mathrm{Tr}[\eta_k\,\eta_{-k}\,, H]\,F \\ \mathrm{Tr}[\eta^+_{-k}\,\eta_k^+\,, H]\,F & \mathrm{Tr}[\eta_k^+\,\eta_k\,, H]\,F \end{pmatrix} \quad . \tag{2.19}$$

The right hand side of eq. (2.18) can also be evaluated explicitly using eqs. (2.9) and (2.10). Equating the result to (2.19) yields

$$i\,\dot{\nu}_k = \mathrm{Tr}[\eta_k^+\,\eta_k\,, H]\,F \tag{2.20}$$

and

$$i(\dot{x}_y\, y_k - x_k\, \dot{y}_k)(1+2\nu_k) \;=\; \mathrm{Tr}[\eta^+_{-k}\,\eta^+_k\,,\,H]\,F \quad . \tag{2.21}$$

Eqs. (2.16), (2.20) and (2.21) determine the time rate of change of the relevant quantities in terms of expectation values of appropriate commutators. They are however clearly not closed equations since there commutators involve the full time-evolution of the field operator. One can, however, obtain closed equations which are formally equivalent to them by expressing mean values in F as functionals of a reduced density $F_0(t)$ which is at any t completely determined by the values, at that time, of just the relevant variables. This is achieved though the use of the projection technique presented below.

III. Projection Technique and Approximation Scheme

In order to evaluate the equations of motion (2.16), (2.20) and (2.21) we start decomposing F in two parts

$$F \;=\; F_0(t) + F'(t) \quad , \tag{3.1}$$

where $F_0(t)$ is the exponential of a one-boson density, which can be conveniently written as

$$F_0 \;=\; \prod_k \frac{1}{1+\nu_k(t)} \left(\frac{\nu_k(t)}{1+\nu_k(t)}\right)^{\eta^+_k(t)\eta_k(t)} \quad . \tag{3.2}$$

$F_0(t)$ has unit trace, so that $F'(t)$ is a traceless correlation density. The next crucial point is to observe that $F_0(t)$ can be written as a time-dependent projection of F, i.e.,

$$F_0 \;=\; \mathbb{P}(t)\,F \qquad , \qquad \mathbb{P}(t)^2 \;=\; \mathbb{P}(t) \quad . \tag{3.3}$$

It is important to keep in mind that $\mathbb{P}$ is an operator acting on a linear space of densities, sometimes called superspace. Such operators are correspondingly called superoperators. The scalar product for any two vectors of this space is defined as

$$(X,Y) \;=\; \mathrm{Tr}(X^+ Y) \quad . \tag{3.4}$$

In order to construct the projector $\mathbb{P}(t)$ we require that, in addition to eqs. (3.3), it satisfies

$$i\,\dot{F}_0(t) \;=\; [\mathbb{P}(t)\,,\,\mathbb{L}]\,F \;=\; [F_0(t)\,,\,H] + \mathbb{P}(t)[H,F] \quad , \tag{3.5}$$

where $\mathbb{L}$ is the superoperator defined as

$$\mathbb{L}\cdot \;=\; [H\,,\,\cdot] \tag{3.6}$$

H being the Hamiltonian of the field. Eq. (3.5) is the Heisenberg picture counterpart of the equation $(\partial_t\,\mathbb{P}(t))\,F = 0$ which has been used to define $\mathbb{P}(t)$ in the Schrödinger Picture[9]. It is possible to prove that conditions (3.3) and (3.4) make $\mathbb{P}(t)$ unique.

The explicit construction of $\mathbb{P}(t)$ is a lenghtly but straightforward algebraic exercise, the relevant steps of which are given in Appendix A. What one obtains is

$$\begin{aligned}\mathbb{P}(t)\cdot \;=\; & \Bigg\{\left(1-\sum_k \frac{\eta^+_k\eta_k-\nu_k}{1+\nu_k}\right)\mathrm{Tr}(\cdot) + \sum_{k_1k_2}\frac{\eta^+_{k_1}\eta_{k_2}-\nu_{k_2}\,\delta_{k_1k_2}}{\nu_{k_2}(1+\nu_{k_2})}\,\mathrm{Tr}(\eta^+_{k_2}\,\eta_{k_1}\,\cdot) \\ & + \sum_k\left[\frac{\eta_k}{\nu_k}\,\mathrm{Tr}(\eta^+_k\,\cdot) + \frac{\eta^+_k}{1+\nu_k}\,\mathrm{Tr}(\eta_k\,\cdot)\right] + \sum_k\Bigg[\frac{\eta_k\,\eta_{-k}}{2\nu_k\,\nu_{-k}}\,\mathrm{Tr}(\eta^+_{-k}\,\eta^+_k\,\cdot) \\ & + \frac{\eta^+_{-k}\,\eta^+_k}{2(1+\nu_{-k})(1+\nu_k)}\,\mathrm{Tr}(\eta_k\,\eta_{-k}\,\cdot)\Bigg]\Bigg\}\,F_0(t) \quad .\end{aligned} \tag{3.7}$$

Using the scalar product (3.4) one can also obtain $\mathbb{P}^+(t)$ which does not coincide with eq. (3.7), i.e., $\mathbb{P}(t)$ is not an orthogonal projection (see Appendix A).

The next step is to obtain a differential equation of $F'(t)$, which follows immediately from eqs. (3.1) and (3.5). It reads

$$\left(i\,\frac{d}{dt} - \mathbb{P}(t)\,\mathbb{L}\right)F'(t) \;=\; \mathbb{Q}(t)\,\mathbb{L}\,F_0(t) \quad , \tag{3.8}$$

where we introduced the complementary projector $\mathbb{Q}(t) = 1-\mathbb{P}(t)$. This equation has the formal solution

$$F'(t) \;=\; \mathbb{G}(t,0)\,F'(0) - i\int dt'\,\mathbb{G}(t,t')\,\mathbb{Q}(t')\,\mathbb{L}\,F_0(t') \quad . \tag{3.9}$$

The first term accounts for initial correlations. In the second term $\mathbb{G}(t,t')$ is the time-ordered Green's Function

$$\mathbb{G}(t,t') = T \exp i \int_{t'}^{t} d\tau\, \mathbb{P}(\tau)\, \mathbb{L} \quad . \tag{3.10}$$

We see thus that $F'(t)$, and therefore also F (see eq. (3.1)), can be formally expressed in terms of $F_0(t')$ (for $t' \leq t$) and of initial correlations $F'(0)$. This allows us also to express the dynamical equations (2.16), (2.20) and (2.21) as traces over functionals of $F_0(t')$ and of the initial correlations. Since, on the other hand, the reduced density $F_0(t')$ is expressed in terms of the relevant variables alone, we see that the resulting equations are now essentially closed equations. Note however that the complicated time-dependence of the field-operators is explicitly probed through the memory effects present in the expression (3.9) for $F'(t)$. Approximations are therefore needed for the actual evaluation of this object.

A systematic expansion scheme for the memory effects has been discussed in ref. [9] in the Schrödinger picture. An important feature of this scheme is that the mean energy is conserved to all orders, i.e.,

$$\frac{\partial}{\partial t} \langle H \rangle_n = 0 \tag{3.11}$$

where

$$\langle H \rangle_n = \mathrm{Tr}\, H\, F_0^{(n)}(t) + \mathrm{Tr}\, H\, F'^{(n)}(t) \quad ,$$

$F_0^{(n)}$ and $F'^{(n)}$ being the approximation of order n to $F_0(t)$ and $F'(t)$ respectively. Here we implement a modified version of the lowest order approximation given in ref. [9]. It consists in approximating the actual time evolution of the field operators, when evaluating memory effects, by the simpler mean-field evolution given by

$$i\, \dot{\eta}_k = [\eta_k\, , H_0(t)] - i\, \dot{A}_k + i(\dot{x}_k^* x_k - \dot{y}_k^* y_k)\, \eta_k - i(\dot{x}_k^* y_k^* - \dot{x}_k^* y_k^*)\, \eta_k^+ \quad . \tag{3.12}$$

The last three terms account for the (explicit) time dependence of the $\eta_k(t)$ related to the shift amplitudes $A_k(t)$ and to the pairing effects. $H_0(t)$ is taken as the effective mean-field hamiltonian

$$H_0 = \mathbb{P}^+ H + \sum_k \eta_k^+ \mathrm{Tr}[\eta_k\, , H]\, F'(t) - \sum_k \eta_k\, \mathrm{Tr}[\eta_k^+\, , H]\, F'(t)$$
$$+ \sum_k \frac{\eta_{-k}^+ \eta_k^+}{2(1+2\nu_k)}\, \mathrm{Tr}[\eta_k \eta_{-k}\, , H]\, F'(t) - \sum_k \frac{\eta_{-k}\, \eta_k}{2(1+2\nu_k)}\, \mathrm{Tr}[\eta_k^+ \eta_{-k}^+\, , H]\, F'(t) \quad . \tag{3.13}$$

The lowest approximation according to ref. [9] corresponds to taking just first term in this expression. The remaining terms, included here, represent correlation contributions to the effective mean-field. Consistently with this approximation, the Green's function (3.10) is also replaced by

$$\mathbb{G}_0(t,t') = T \exp i \int_{t'}^{t} d\tau\, \mathbb{P}(\tau)\, \mathbb{L}_0(\tau) \tag{3.14}$$

where

$$\mathbb{L}_0 \cdot = [H_0\, , \cdot\,] \tag{3.15}$$

so that the correlation density is written as

$$F'(t) \cong \mathbb{G}_0(t,t')\, F'(0) - i \int_0^t dt'\, \mathbb{G}_0(t,t')\, \mathbb{Q}(t')\, \mathbb{L}\, F_0(t') =$$
$$= \mathbb{G}_0(t,t')\, F'(0) - i \int_0^t dt'\, \mathbb{Q}(t')\, \mathbb{L}\, F_0(t') \quad , \tag{3.16}$$

since it is easy to see that eqs. (3.12) and (3.14) imply that $\mathbb{G}_0$ acts as the unit operator in the memory integral.

According to the approximation scheme just described the basic dynamical equations to be solved are eqs. (2.16), (2.20) and (2.21), where F is expressed in terms of eqs. (3.2) and (3.16). Furthermore, for the purpose of evaluating eq. (3.16) the field operators are time-evolved according to eq. (3.12). The resulting scheme can be interpreted as follows. The dynamical evolution of the field is splitt into a pure mean-field part, related to the contributions to the dynamical equations involving the projected density $F_0(t)$, plus correlation contributions, approximated by the contributions involving the adopted form for $F'(t)$. These are nonunitary, in the sense that they change the coherence properties of $F_0(t)$ through the time-evolution of the occupation numbers $\nu_k(t)$ (see eq. (2.20)). In fact, replacing F by just $F_0(t)$ in this equation gives $\dot{\nu}_k(t) = 0$. Consequently the entropy function associated with $F_0(t)$ will change in time as a result of the correlation contributions, which therefore perform as collision terms from the point of view of the one-boson density. Moreover, the correlation contributions will also modify the pure mean-field evolution in eqs. (2.16) and (2.21). The adopted approximation ammounts to restricting correlation effects to second order in H (as can be seen by substituting eq. (3.15) in the dynamical equations) while taking full account of the effective mean-field (see eqs. (3.12) and (3.13)).

IV. Mean Field Approximation: Renormalization and Effective Potential

We now discuss the actual evaluation of the general expressions obtained in the preceding sections for the Hamiltonian

$$H = \int dx\, \mathcal{H} \tag{4.1}$$

$$\mathcal{H} = \frac{\pi^2}{2} + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{\delta m^2}{2}\phi^2 + \lambda\phi \ . \tag{4.2}$$

In this section we consider only the lowest (mean-field) approximation, which amounts to assuming $F'(t) \equiv 0$. Collisional corrections will be discussed in section V.

In eq. (4.2), m stands for the renormalized mass, and a prescription for the mass counterterm δm^2 will be given below. The last term is an external linear coupling which will be used to allow for constraining the expectation value of ϕ at equilibrium in an evaluation of the effective potential.

Consistently with the orthogonality and completeness relations (2.13) and (2.14) we parametrize the elements of the transformation matrix $\mathbf{X}_k$ (see eq. (2.15)) as

$$x_k = \mathrm{ch}\,\sigma_k + \frac{i}{2}\tau_k \tag{4.3}$$

$$y_k = \mathrm{sh}\,\sigma_k + \frac{i}{2}\tau_k \tag{4.4}$$

with σ_k and τ_k real. It is also convenient to associate σ_k with a dynamic mass parameter $\mu_k(t)$ through

$$\sqrt{k^2+\mu_k^2(t)} = e^{2\sigma_k(t)}\sqrt{k^2+\mu^2} \tag{4.5}$$

as shown below in eq. (4.8), τ_k is related to $\dot{\mu}_k(t)$. This $\mu_k(t)$ can be seen as an effective mass incorporating momentum-dependent mean-field interaction of the uniform system.

The mean-field approximation to the dynamical equations (2.16), (2.20) and (2.21) amounts to replacing F by just $F_0(t)$. Introducing the ingredients described above one obtains, assuming uniform condensates only, i.e., $\Gamma_k(t) = \Gamma_0(t)\,\delta_{k0}$

$$\partial_t^2\langle\phi\rangle = -\left(\lambda + m^2\langle\phi\rangle + \frac{g}{3!}\langle\phi\rangle^3\right) - \frac{g}{4L}\langle\phi\rangle\sum_k\frac{1}{\sqrt{k^2+\mu_k^2}} - \delta m^2\langle\phi\rangle\ ; \tag{4.6}$$

$$\dot{\nu}_k = 0 \quad ; \tag{4.7}$$

$$\dot{\mu}_k = -2(k^2+\mu^2)^{1/4}\cdot\frac{(k^2+\mu_k^2)^{5/4}}{\mu_k}\,\tau_k \quad ; \tag{4.8}$$

$$\left(\frac{k^2+\mu^2}{k^2+\mu_k^2}\right)^{1/4}\left(\frac{1}{2}\frac{\mu_k\dot{\mu}_k}{k^2+\mu_k^2}\tau_k + \dot{\tau}_k\right) = \sqrt{k^2+\mu_k^2} - \sqrt{k^2+\mu^2}\,\tau_k^2$$
$$- \frac{k^2+m^2}{\sqrt{k^2+\mu_k^2}} - \frac{g}{2}\frac{\langle\phi\rangle^2}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+\mu_k^2}}\left[\frac{g}{4L}\sum_{k'}\frac{1}{\sqrt{k'^2+\mu_{k'}^2}} + \delta m^2\right]. \tag{4.9}$$

Eq. (4.7), in particular, shows that the reduced occupation numbers ν_k are constant in the mean-field approximation.

It is interesting to look at the static solutions $\langle\phi\rangle = \phi_0$ of the mean-field equations. They are given as the solutions of

$$\lambda + m^2\phi_0 + \frac{g}{3!}\phi_0^3 + \frac{g\phi_0}{4L}\sum_{k'}\frac{1}{\sqrt{k'^2+\mu_{k'}^2}} + \delta m^2\phi_0 = 0 \tag{4.10}$$

and

$$\mu_k^2 = m^2 + \frac{g}{2}\phi_0^2 + \frac{g}{4L}\sum_{k'}\frac{1}{\sqrt{k'^2+\mu_{k'}^2}} + \delta m^2 \quad . \tag{4.11}$$

Eq. (4.11) shows that μ_k^2 is in fact independent of k in the static case. The logarithmic divergences of the sums in eqs. (4.10) and (4.11) are controlled by adjusting the mass counterterm as

$$\delta m^2 = -\frac{g}{4L}\sum_k\frac{1}{\sqrt{k^2+m^2}} \quad . \tag{4.12}$$

The mean-field energy density, on the other hand, is easily evaluated as

$$\begin{aligned}\left\langle\frac{E}{L}\right\rangle &= \frac{1}{L}\,\mathrm{Tr}\,H\,F_0 = \frac{\langle\pi\rangle^2}{2} + \lambda\langle\phi\rangle + \frac{m^2}{2}\langle\phi\rangle^2 + \frac{g}{4!}\langle\phi\rangle^4 \\ &+ \frac{g\langle\phi\rangle^2}{8L}\sum_k\left(\frac{1}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+m^2}}\right) + \frac{g}{32L^2}\left[\sum_k\left(\frac{1}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+m^2}}\right)\right]^2 \\ &- \frac{g}{32L^2}\left(\sum_k\frac{1}{\sqrt{k^2+m^2}}\right)^2 + \frac{1}{2L}\sum_k\left(\sqrt{k^2+\mu_k^2} - \frac{m^2-\mu_k^2}{2\sqrt{k^2+\mu_k^2}}\right) \\ &+ \frac{1}{4L}\sum_k\sqrt{k^2+\mu^2}\,\tau_k^2\end{aligned} \tag{4.13}$$

which is rendered finite after substracting a (divergent) vacuum energy (cf. ref. [11]). It is straightforward to check that this energy density is conserved under the mean-field equations of motion (4.6)–(4.9).

The mean-field effective potential $V_{\text{eff}}(\phi_0)$ is now easily obtained from eq. (4.13) evaluated in the static case. As shown by eq. (4.10), the equilibrium value $\langle\phi\rangle$ can

be adjusted through the external coupling parameter λ (which acts as a Lagrange multiplier) so that

$$V_{\text{eff}}(\phi_0) = \left\langle\frac{E}{L}\right\rangle_{\langle\phi\rangle=\phi_0} - \lambda\phi_0 - \text{vacuum energy density} \tag{4.14}$$

yielding, in the continuum limit $(L\to\infty)$

$$V_{\text{eff}}(\phi_0) = \frac{m^2}{2}\phi_0^2 + \frac{g}{4!}\phi_0^4 + \frac{g}{16\pi}\phi_0^2\ln\frac{m^2}{\mu^2} + \frac{g}{128\pi}\left(\ln\frac{m^2}{\mu^2}\right)^2 + \frac{1}{8\pi}(\mu^2-m^2) + \frac{m^2}{8\pi}\ln\frac{m^2}{\mu^2} \tag{4.15}$$

which reproduces the well known effective potential obtained in the Gaussian approximation [11].

V. Collisional Dynamics for Homogeneous Systems

In order to calculate the collision terms of equations (2.16), (2.20) and (2.21), one must evaluate traces of the type

$$\mathrm{Tr}[\hat{O}(t)\,,\,H]\int_0^t dt'\,\mathbb{Q}(t')[H\,,\,F_0(t')] \tag{5.1}$$

where $\hat{O}(t)$ can be η, $\eta^+\eta$ or $\eta\eta$. The density $\mathbb{Q}(t')[H\,,\,F_0(t')]$ can be evaluated in straightforward way using eqs. (3.7), (4.2) and (3.2). One obtains

$$\begin{aligned}\mathbb{Q}(t')[H\,,\,F_0(t')] &= \frac{g}{96L}\sum_{k_1k_2k_3k_4}\frac{e^{-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}}}{\sqrt{k_{01}k_{02}k_{03}k_{04}}}\,\delta_{k_1+k_2+k_3+k_4\,,0} \\ &\times\left[\frac{\eta_{k_1}^+\eta_{k_2}^+\eta_{k_3}^+\eta_{k_4}^+}{(1+\nu_{k_1})(1+\nu_{k_2})(1+\nu_{k_3})(1+\nu_{k_4})} - \frac{\eta_{k_1}\eta_{k_2}\eta_{k_3}\eta_{k_4}}{\nu_{k_1}\nu_{k_2}\nu_{k_3}\nu_{k_4}}\right]_{t'} \\ &\times\left(1+\sum_i^4\nu_{k_i} + \sum_{i<j}^4\nu_{k_i}\nu_{k_j} + \sum_{i<j<l}^4\nu_{k_i}\nu_{k_j}\nu_{k_l}\right)_{t'} F_0(t') \\ &+ \frac{g}{24L}\sum_{k_1k_2k_3k_4}\frac{e^{-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}}}{\sqrt{k_{01}k_{02}k_{03}k_{04}}}\,\delta_{k_1+k_2+k_3-k_4\,,0}\end{aligned}$$

$$
\times \left[\frac{\eta^+_{k_1}\,\eta^+_{k_2}\,\eta^+_{k_3}\,\eta_{k_4}}{(1+\nu_{k_1})(1+\nu_{k_2})(1+\nu_{k_3})\nu_{k_4}} - \frac{\eta^+_{k_4}\,\eta_{k_1}\,\eta_{k_2}\,\eta_{k_3}}{\nu_{k_1}\,\nu_{k_2}\,\nu_{k_3}(1+\nu_{k_4})} \right]_{t'}
$$

$$
\times \left[\nu_{k_4} - \nu_{k_1}\,\nu_{k_2}\,\nu_{k_3} + \nu_{k_4} \times \left(\sum_i^3 \nu_{k_i} \right) + \nu_{k_4} \left(\sum_{i<j}^3 \nu_{k_i}\,\nu_{k_j} \right) \right]_{t'} F_0(t')
$$

$$
+ \frac{g}{16L} \sum_{k_1 k_2 k_3 k_4} \frac{\mathrm{e}^{-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}}}{\sqrt{k_{01}k_{02}k_{03}k_{04}}}\, \delta_{k_1+k_2-k_3-k_4\,,0}
$$

$$
\times \left(1 - \frac{\nu_{k_1}}{1+\nu_{k_1}}\,\frac{\nu_{k_2}}{1+\nu_{k_2}}\,\frac{1+\nu_{k_3}}{\nu_{k_3}}\,\frac{1+\nu_{k_4}}{\nu_{k_4}} \right)_{t'} \eta^+_{k_1}\,\eta^+_{k_2}\,\eta_{k_3}\,\eta_{k_4}\,F_0(t')
$$

$$
+ \frac{g}{12\sqrt{2L}}\,\langle\phi\rangle_{t'} \sum_{k_1 k_2 k_3} \frac{\mathrm{e}^{-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3})_{t'}}}{\sqrt{k_{01}k_{02}k_{03}}}\, \delta_{k_1+k_2+k_3\,,0}
$$

$$
\times \left[\frac{\eta^+_{k_1}\,\eta^+_{k_2}\,\eta^+_{k_3}}{(1+\nu_{k_1})(1+\nu_{k_2})(1+\nu_{k_3})} - \frac{\eta_{k_1}\,\eta_{k_2}\,\eta_{k_3}}{\nu_{k_1}\,\nu_{k_2}\,\nu_{k_3}} \right]_{t'} \left(1 + \sum_i^3 \nu_{k_i} + \sum_{i<j}^3 \nu_{k_i}\,\nu_{k_j} \right)_{t'} F_0(t')
$$

$$
+ \frac{g}{4\sqrt{2L}}\,\langle\phi\rangle_{t'} \sum_{k_1 k_2 k_3} \frac{\mathrm{e}^{-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3})_{t'}}}{\sqrt{k_{01}k_{02}k_{03}}}\, \delta_{k_1+k_2+k_3\,,0}
$$

$$
\times \left[\frac{\eta^+_{k_1}\,\eta^+_{k_2}\,\eta_{k_3}}{(1+\nu_{k_1})(1+\nu_{k_2})\nu_{k_3}} - \frac{\eta^+_{k_1}\,\eta_{k_2}\,\eta_{k_3}}{\nu_{k_1}\,\nu_{k_2}\,(1+\nu_{k_3})} \right]_{t'} \left[\nu_{k_3}\,(1+\nu_{k_1}+\nu_{k_2}) - \nu_{k_1}\,\nu_{k_2} \right]_{t'} F_0(t')
$$

$$
+ \frac{g}{4L} \sum_{k_1 k_2} \frac{\mathrm{e}^{-2(\sigma_{k_1}+\sigma_{k_2})_{t'}}}{k_{01}\,k_{02}} \left[\frac{\eta^+_{-k_2}\,\eta^+_{k_2}}{(1+\nu_{k_1})(1+\nu_{k_2})} - \frac{\eta_{k_2}\,\eta_{-k_2}}{\nu_{k_1}\,\nu_{k_2}} \right]_{t'} \nu_{k_1}\,(1+\nu_{k_1}+\nu_{k_2})_{t'}\,F_0(t')
$$

$$
- \frac{g\langle\phi\rangle_{t'}}{\sqrt{8L\mu}}\,\mathrm{e}^{-\sigma_0} \sum_k \frac{\mathrm{e}^{-2\sigma_k(t')}\,\nu(t')}{k_0} \left(\frac{\eta^+_0}{1+\nu_0} - \frac{\eta_0}{\nu_0} \right)_{t'} F_0(t') \quad . \tag{5.2}
$$

The traces in eq. (5.1) still cannot be taken directly, since the operators in eq. (5.2) and in the first commutator are at different times. To overcome this we adopt the approximation discussed in section III and describe the time evolution of the operators by eq. (3.12). Using also equations (3.13), (4.3) and (4.4) one obtains

$$
i\dot{\eta}_k = \left\{ \left[\frac{k^2+m^2}{2} + \frac{k^2+\mu_k^2}{2} + \frac{1}{2}\sqrt{k^2+\mu^2}\,\sqrt{k^2+\mu_k^2}\,\tau_k^2 + \frac{g}{4}\,\langle\phi\rangle^2 \right.\right.
$$

$$
+ \frac{g}{4L} \sum_{k'} \frac{\nu_{k'}}{\sqrt{k'^2+\mu_{k'}^2}} + \frac{g}{8L} \sum_{k'} \left(\frac{1}{\sqrt{k'^2+\mu_{k'}^2}} - \frac{1}{\sqrt{k^2+m_k^2}} \right) \Bigg] \frac{1}{\sqrt{k^2+\mu_k^2}}
$$

$$
+ \frac{1}{2} \left(\frac{k^2+\mu^2}{k^2+\mu_k^2} \right)^{1/4} \left(\frac{1}{2}\,\frac{\mu_k\,\dot{\mu}_k}{k^2+\mu_k^2}\,\tau_k + \dot{\tau}_k \right) \Bigg\}\,\eta_k = f_k(t)\,\eta_k \quad . \tag{5.3}
$$

The operators η_k at different times are thus related as

$$
\eta_k(t) = \mathrm{e}^{i\varphi_k(t,t')}\,\eta_k(t') \tag{5.4}
$$

the phase $\varphi_k(t,t')$ being given by

$$
\varphi_k(t,t') = -\int_{t'}^{t} d\tau\, f_k(\tau) \quad . \tag{5.5}
$$

The derivation of the proposed approximation to the collisional dynamics is now a lengthly but straightforward algebraic exercise. The resulting equations of motion are

$$
\partial_t^2\langle\phi\rangle = -m^2\langle\phi\rangle - \frac{g}{3!}\,\langle\phi\rangle^3 - \frac{g}{4L}\,\langle\phi\rangle \sum_k \left[\frac{1}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+m^2}} \right]
$$

$$
- \frac{g}{2L}\,\langle\phi\rangle \sum_k \frac{\nu_k}{\sqrt{k^2+\mu_k^2}} - \Gamma_{\langle\phi\rangle}(t) \quad , \tag{5.6}
$$

$$
\dot{\nu}_k = \Gamma_\nu(t) \quad , \tag{5.7}
$$

$$
\dot{\mu}_k = -2(k^2+\mu^2)^{1/4}\,\frac{(k^2+\mu_k^2)^{5/4}}{\mu_k}\,\tau_k + \frac{2(k^2+\mu_k^2)\,\dot{\nu}_k}{(1+2\nu_k)\,\mu_k} \quad , \tag{5.8}
$$

$$
\left(\frac{k^2+\mu^2}{k^2+\mu_k^2} \right)^{1/4} \left(\frac{1}{2}\,\frac{\mu_k\,\dot{\mu}_k}{k^2+\mu_k^2}\,\tau_k + \dot{\tau}_k \right) = \sqrt{k^2+\mu_k^2} - \sqrt{k^2+\mu_k^2}\,\tau_k^2
$$

$$
- \frac{k^2+m^2}{\sqrt{k^2+\mu_k^2}} - \frac{g}{2}\,\frac{\langle\phi\rangle^2}{\sqrt{k^2+\mu_k^2}} - \frac{g}{2L}\,\frac{1}{\sqrt{k^2+\mu_k^2}} \sum_{k'} \frac{\nu_{k'}}{\sqrt{k'^2+\mu_{k'}^2}}
$$

$$
- \frac{g}{4L}\,\frac{1}{\sqrt{k^2+\mu_k^2}} \sum_{k'} \left[\frac{1}{\sqrt{k'^2+\mu_{k'}^2}} - \frac{1}{\sqrt{k'^2+m^2}} \right] - \frac{\Gamma_\mu(t)}{1+2\nu_k} \quad . \tag{5.9}
$$

Where the collision integrals $\Gamma(t)$ are

$$\Gamma_{\langle\phi\rangle}(t) = \frac{g^2}{24L^2} \sum_{k_1k_2k_3} \left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k_3,0}\, I^{(4)}_{k_1k_2k_3} + 3\delta_{k_1+k_2-k_3,0}\, I^{(5)}_{k_1k_2k_3}\right) \quad , \tag{5.10}$$

$$\Gamma_{\nu}(t) = \frac{g^2}{48L^2} \sum_{k_1k_2k_3} \left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}} \frac{1}{\sqrt{k^2+\mu_{k}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k_3+k,0}\, J^{(1)}_{k_1k_2k_3k} + 3\delta_{k_1+k_2+k-k_3,0}\, J^{(2)}_{k_1k_2kk_3}\right.$$

$$\left. - \delta_{k_1+k_2+k_3-k,0}\, J^{(2)}_{k_1k_2k_3k} + 3\delta_{k+k_1-k_2-k_3,0}\, J^{(3)}_{kk_1k_2k_3}\right)$$

$$+ \frac{g^2}{8L}\langle\phi\rangle \sum_{k_1k_2} \left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k^2+\mu_{k}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k,0}\, J^{(4)}_{k_1k_2k} + 2\delta_{k_1+k-k_2,0}\, J^{(5)}_{k_1kk_2} - \delta_{k_1+k_2-k,0}\, J^{(5)}_{k_1k_2k}\right) \quad , \tag{5.11}$$

$$\Gamma_{\mu}(t) = \frac{g^2}{24L^2} \sum_{k_1k_2k_3} \left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}} \frac{1}{\sqrt{k^2+\mu_{k}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k_3+k,0}\, I^{(1)}_{k_1k_2k_3k} + 3\delta_{k_1+k_2+k-k_3,0}\, I^{(2)}_{k_1k_2kk_3}\right.$$

$$\left. + \delta_{k_1+k_2+k_3-k,0}\, I^{(2)}_{k_1k_2k_3k} + \delta_{k+k_1-k_2-k_3,0}\, I^{(3)}_{kk_1k_2k_3}\right)$$

$$+ \frac{g^2}{4L}\langle\phi\rangle \sum_{k_1k_2} \left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k^2+\mu_{k}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k,0}\, I^{(4)}_{k_1k_2k} + 2\delta_{k_1+k-k_2,0}\, I^{(5)}_{k_1kk_2} + \delta_{k_1+k_2-k,0}\, I^{(5)}_{k_1k_2k}\right) \quad . \tag{5.12}$$

The density energy is

$$\left\langle\frac{E}{L}\right\rangle = \frac{\langle\Pi\rangle^2}{2} + \frac{m^2}{2}\langle\phi\rangle^2 + \frac{g}{4!}\langle\phi\rangle^4 + \frac{g}{8L}\langle\phi\rangle^2 \sum_k \left[\frac{1}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+m^2}}\right]$$

$$+ \frac{g}{4L}\langle\phi\rangle^2 \sum_k \frac{\nu_k}{\sqrt{k^2+\mu_k^2}} + \frac{g}{8L^2}\sum_{k_1}\frac{\nu_{k_1}}{\sqrt{k_1^2+\mu_{k_1}^2}} \sum_{k_2}\left[\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} - \frac{1}{\sqrt{k_2^2+m^2}}\right]$$

$$+ \frac{g}{8L^2}\left[\sum_k \frac{\nu_k}{\sqrt{k^2+\mu_k^2}}\right]^2 + \frac{g}{32L^2}\left[\sum_k\left(\frac{1}{\sqrt{k^2+\mu_k^2}} - \frac{1}{\sqrt{k^2+m^2}}\right)\right]^2$$

$$+ \frac{1}{4L}\sum_k\left[\frac{k^2+m^2}{\sqrt{k^2+\mu_k^2}} + \sqrt{k^2+\mu_k^2} + \sqrt{k^2+\mu^2}\,\tau_k^2\right] - \frac{1}{2L}\sum_k\sqrt{k^2+m^2}$$

$$+ \frac{g^2}{192L^3}\sum_{k_1k_2k_3k_4}\left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}} \frac{1}{\sqrt{k_4^2+\mu_{k_4}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k_3+k_4,0}\, I^{(1)}_{k_1k_2k_3k_4} + 4\delta_{k_1+k_2+k_3-k_4,0}\, I^{(2)}_{k_1k_2k_3k_4} + 3\delta_{k_1+k_2-k_3-k_4,0}\, I^{(3)}_{k_1k_2k_3k_4}\right)$$

$$+ \frac{g^2}{24L^2}\langle\phi\rangle\sum_{k_1k_2k_3}\left[\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}} \frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}} \frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right]^{1/2}$$

$$\times \left(\delta_{k_1+k_2+k_3,0}\, I^{(4)}_{k_1k_2k_3} + \delta_{k_1+k_2-k_3,0}\, I^{(5)}_{k_1k_2k_3}\right) \quad . \tag{5.13}$$

In these equations use was made of the abbreviations

$$I^{(1)}_{k_1k_2k_3k_4}(t) = \int_0^t dt' \prod_{i=1}^4 \left[\frac{1}{\sqrt{k_i^2+\mu_{k_1}^2}}\right]^{1/2}_{t'}$$

$$\times \left(1 + \sum_i^4 \nu_{k_i} + \sum_{i<j}^4 \nu_{k_i}\nu_{k_j} + \sum_{i<j<l}^4 \nu_{k_i}\nu_{k_j}\nu_{k_l}\right)_{t'}$$

$$\times \ \mathrm{sen}[\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t') + \varphi_{k_3}(t,t') + \varphi_{k_4}(t,t')] \quad , \tag{5.14}$$

$$I^{(2)}_{k_1k_2k_3k_4}(t) = \int_0^t dt' \prod_{i=1}^{4} \left[\frac{1}{\sqrt{k_i^2+\mu_{k_1}^2}}\right]^{1/2}_{t'}$$

$$\times \left(\nu_{k_4} - \nu_{k_1}\nu_{k_2}\nu_{k_3} + \nu_{k_4}\sum_i^3 \nu_{k_i} + \nu_{k_4}\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right)_{t'}$$

$$\times \ \mathrm{sen}[\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t') + \varphi_{k_3}(t,t') - \varphi_{k_4}(t,t')] \ , \qquad (5.15)$$

$$I^{(3)}_{k_1k_2k_3k_4}(t) = \int_0^t dt' \prod_{i=1}^{4} \left[\frac{1}{\sqrt{k_i^2+\mu_{k_1}^2}}\right]^{1/2}_{t'}$$

$$\times \ [\nu_{k_3}\nu_{k_4}(1+\nu_{k_1})(1+\nu_{k_2}) - \nu_{k_1}\nu_{k_2}(1+\nu_{k_3})(1+\nu_{k_4})]_{t'}$$

$$\times \ \mathrm{sen}[\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t') - \varphi_{k_3}(t,t') - \varphi_{k_4}(t,t')] \ , \qquad (5.16)$$

$$I^{(4)}_{k_1k_2k_3}(t) = \int_0^t dt' \prod_{i=1}^{3} \left[\frac{1}{\sqrt{k_i^2+\mu_{k_1}^2}}\right]^{1/2}_{t'} \langle\phi\rangle_{t'} \left(1 + \sum_i^3 \nu_{k_i} + \sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right)_{t'}$$

$$\times \ \mathrm{sen}[\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t') + \varphi_{k_3}(t,t')] \ , \qquad (5.17)$$

$$I^{(5)}_{k_1k_2k_3}(t) = \int_0^t dt' \prod_{i=1}^{3} \left[\frac{1}{\sqrt{k_i^2+\mu_{k_1}^2}}\right]^{1/2}_{t'} \langle\phi\rangle_{t'} \ [\nu_{k_3}(1+\nu_{k_1}+\nu_{k_2}) - \nu_{k_1}\nu_{k_2}]_{t'}$$

$$\times \ \mathrm{sen}[\varphi_{k_1}(t,t') + \varphi_{k_2}(t,t') - \varphi_{k_3}(t,t')] \ , \qquad (5.18)$$

The $J^{(i)}$ are identical to $I^{(i)}$ with the sine functions replaced by cosines in the integrand. Energy conservation $\partial_t \left\langle \frac{E}{L} \right\rangle = 0$ can also be checked directly by using the dynamical equations.

VI. Numerical Results and Concluding Remarks

In this section, we give numerical solutions of the equations of motion (5.6)–(5.9). A useful technique to treat the memory integrals in these equations is described in appendix B. In order to control the domain of validity of the approximations involved in the derivation of the equations of motions (see sections IV and V), it is useful to inspect also corresponding solutions for quantum mechanics (i.e., $0+1$ dimensions)[12]. We find, by comparison with the exact numerical solution which are available in this case, that the collisional approximation improves qualitatively the dynamical description of field variables.

In what follows we use natural units. As a first case, we take the parameters of the hamiltonian (4.2) as $m = 1.2$, $g = 2$. The static mean-field solution is then $\langle\pi\rangle = 0$, $\langle\phi\rangle = 0$, $\nu_k = 0$ and $\mu_k = \mu = m$. Figures 1 show the mean-field and the collisional approximations to the time evolution of the various dynamical variables. for the initial condition $f'(0) = 0$, $\langle\phi\rangle(0) = 1$, $\langle\pi\rangle(0) = 0$, $\nu_k(0) = 0$ and $\mu_k(0) = \mu$. Periodic boundary conditions were implemented as $k = \frac{2\pi}{L}N$ with $L = 40$ and $-6 \leq N \leq 6$. A comparison with a calculation involving a larger dimensionality indicated substantial convergence for the variables shown. Although the mean-field and collisional approximations to $\langle\phi\rangle(t)$, Fig. 1A, do not differ much, the former does not show the damping which is present in collisional approximation. More dramatic difference show up however in the case of the time evolution of $\nu_k(t)$ and $\mu_k(t)$, as shown in Figs. 1B and 1C. A natural way to interpret these results is that $\nu_k = 0$ constraint imposed by the mean-field approximation strongly distorts the dynamical behavior of the extended density $\mathcal{R}_k$, as revealed by $\mu_k(t)$. This effect can be noted also in the results for $0+1$ dimensions as shown in the Figures 2A–2C. The exact numerical solution is also shown in this case. It shows in fact that the collisional effects are necessary to describe properly the dynamics of $\mu_k(t)$. However, the collisional approximation is seen here also to fail for large enough times, leading in particular to an overestimation of $\nu_k(t)$. Numerical checks in the code (involving e.g. energy conservation) demonstrate that this is not due to numerical failures, but should be seen as a limitation of the adopted approximation for the collision terms.

Figures 3 refer to results in the case of broken-symmetry and initial conditions $\langle\phi\rangle(0)=0$, $\langle\pi\rangle(0)=0$, $\nu_k(0)=0$, $\mu_k(0)=0{,}08$, and $\mu_{k\neq 0}(0)=0{,}05=\mu$. The k-sum is now cut-off at $|N|=12$ with $L=420$ which indicated sufficient convergence. Figures 3A and 3B show dramatic collisional effects on the time evolution of $\mu_k(t)$: in fact $\mu_k(t)$ initially decreases in the mean-field approximation while it increases when collisional terms are turned on. In the mean-field approximation, the $\mu_k(t)$ are the only degrees of freedom affecting the root-mean-square field (evaluated here simply as $\langle\phi^2\rangle-\langle\phi\rangle^2=\sum_k \frac{1+2\nu_k}{k^2+\mu_k^2}$), and it is natural to expect that they approach the equilibrium value, $\mu=0{,}05$. In the collisional calculation, the root-mean-square field evolves also due to the change in time of the occupation $\nu_k(t)$. We see again, therefore, that the mean-field constraint $\nu_k=0$ strongly distorts the dynamical behavior of mass parameters. Figure 3E shows the root-mean-square field $\langle\phi(t)^2\rangle^{\frac{1}{2}}$ as a function of time. It shows that the increase of $\mu_k(t)$ is overcompensated by the change of the $\nu_k(t)$, resulting in the positive evolution $\sqrt{\langle\phi^2\rangle}$ in the case of collisional approximation. Figures 4A–C show the corresponding results in 0+1 dimensions. It can be seen that again the collisional approximation is able to reproduce qualitatively the exact time evolution of ν_t and μ_t, until it fails again due to overestimation of ν_t for larger times.

We conclude, from these examples, that the mean-field approximation fails qualitatively in the description of the variables of field. These failures are partially corrected by the collisional integrals used here. However, improvements of the simplest approximation to the collisional effects, as implemented here, are needed if one wishes a quantitatively reliable description for larger times. Attempts along this line are under way.

Finally, we comment on the extension of our treatment to non-uniform field configurations. In this case, the spatial dependence of the field operator is expanded

in the general natural orbitals of the extended density (2.7). These orbitals can be given in terms of a momentum expansion which will also evolve in time according to additional dynamical equations which are in this case obtained from the Heisenberg equation of motion for $\phi(x)$, again in the close analogy with the non-relativistic many body treatment. Further details on this point are given elsewhere[8].

Appendix A: Construction of the Projectors $\mathbb{P}$ and $\mathbb{P}^+$.

In order to simplify the presentation, we shall show the technique for the case of $0+1$ dimensions. The same general procedure applies also to the case of $1+1$ or higher dimensions.

In section II, we have stated the conditions to be fulfilled by $\mathbb{P}$ as

$$F_0(t) = \mathbb{P}(t)\,F \tag{A.1}$$

$$\mathbb{P}(t)\,\mathbb{P}(t) = \mathbb{P}(t) \tag{A.2}$$

$$i\,\dot{F}_0(t) = [F_0, H] + \mathbb{P}(t)[H, F] \tag{A.3}$$

where $F_0(t)$ in $0+1$ dimensions is

$$F_0(t) = \frac{1}{1+\nu(t)}\left(\frac{\nu(t)}{1+\nu(t)}\right)^{\eta^+(t)\eta(t)} . \tag{A.4}$$

The derivative of $F_0(t)$ with respect to time is first written as

$$i\,\dot{F}_0 = \frac{\eta^+\eta-\nu}{\nu(1+\nu)}\,F_0\,\mathrm{Tr}\,\eta^+\eta[H,F] + \frac{i}{1+\nu}\,\frac{d'}{dt}\left(\frac{\nu}{1+\nu}\right)^{\eta^+\eta} \tag{A.5}$$

where in the last term $\frac{d'}{dt}$ acts only on the operators η and η^+. In order to evaluate this term, rewrite the exponential as

$$\left(\frac{\nu(t)}{1+\nu(t)}\right)^{\eta^+(t)\eta(t)} = e^{m(t)\eta^+(t)\eta(t)} \tag{A.6}$$

so that

$$i\,\frac{d'}{dt}\,e^{m\eta^+\eta} = \sum_{n=0}^{\infty}\frac{m^n}{n!}\left[i\left(\frac{d}{dt}\,\eta^+\eta\right)(\eta^+\eta)^{n-1} + \cdots + (\eta^+\eta)^{n-1}\,i\,\frac{d}{dt}\,(\eta^+\eta)\right] . \tag{A.7}$$

Using eq. (4.5) and the Heisenberg equation of motion one finds

$$i\,\frac{d}{dt}\,\eta^+\eta = [\eta^+\eta\,,\,H] - i\,\dot{A}^*\,\eta - i\,\dot{A}\,\eta - i(\dot{x}y - x\dot{y})\,\eta\eta - i(\dot{x}^*y^* - x^*\dot{y}^*)\,\eta^+\eta^+ \;, \tag{A.8}$$

so that the last term of (A.5) becomes

$$\begin{aligned}\frac{i}{1+\nu}\,\frac{d'}{dt}\,e^{m\eta^+\eta} &= \frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\left\{[\eta^+\eta\,,\,H](\eta^+\eta)^{n-1} + \cdots + (\eta^+\eta)^{n-1}[\eta^+\eta\,,\,H]\right\}\\ &- \frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\,\dot{A}\left\{i\,\eta(\eta^+\eta)^{n-1} + \cdots + (\eta^+\eta)^{n-1}i\,\eta\right\}\\ &- \frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\,\dot{A}\left\{i\,\eta^+(\eta^+\eta)^{n-1} + \cdots + (\eta^+\eta)^{n-1}i\,\eta^+\right\}\\ &- \frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\,(\dot{x}y - x\dot{y})\left\{i\eta\eta(\eta^+\eta)^{n-1} + \cdots + (\eta^+\eta)^{n-1}i\eta\eta\right\}\\ &- \frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\,(\dot{x}^*y^* - x^*\dot{y}^*)\left\{i\,\eta + \eta^+(\eta^+\eta)^{n-1} + \cdots + i(\eta^+\eta)^{n-1}\eta^+\eta^+\right\} .\end{aligned} \tag{A.9}$$

In order to reobtain the exponential (A.6), one moves the operators η, η^+, $\eta\eta$, $\eta^+\eta^+$, to the right and obtain, after some algebra

$$\begin{aligned}\frac{1}{1+\nu}\,\frac{d'}{dt}\,e^{m\eta^+\eta} &= [F_0, H] + i\,\dot{A}^*\,\frac{\eta}{\nu}\,F_0 + i\,\dot{A}\,\frac{\eta^+}{1+\nu}\,F_0 + i(\dot{x}y - x\dot{y})(1+2\nu)\,\frac{\eta\eta}{2\nu^2}\,F_0\\ &+ i(\dot{x}^*y^* - x^*\dot{y}^*)(1+2\nu)\,\frac{\eta^+\eta^+}{2(1+\nu)^2}\,F_0 .\end{aligned} \tag{A.10}$$

Using the dynamic equations for the extended density, eqs. (2.20) and (2.21), one gets

$$\begin{aligned}i\,\dot{F}_0 &= \frac{\eta^+\eta-\nu}{\nu(1+\nu)}\,F_0\,\mathrm{Tr}\,\eta^+\eta[H,F] + [F_0, H] + \frac{\eta}{\nu}\,F_0\,\mathrm{Tr}\,\eta^+[H,F]\\ &+ \frac{\eta^+}{1+\nu}\,F_0\,\mathrm{Tr}\,\eta[H,F] + \frac{\eta\eta}{2\nu^2}\,F_0\,\mathrm{Tr}\,\eta^+\eta^+[H,F] + \frac{\eta^+\eta^+}{2(1+\nu)^2}\,F_0\,\mathrm{Tr}\,\eta\eta[H,F] .\end{aligned} \tag{A.11}$$

Eq. (A.3) is therefore satisfied by

$$\bar{P}\cdot \;=\; \frac{\eta^+\eta-\nu}{\nu(1+\nu)}\,F_0\,\mathrm{Tr}(\eta^+\eta\,\cdot)+\frac{\eta}{\nu}\,F_0\,\mathrm{Tr}\,(\eta^+\,\cdot)+\frac{\eta^+}{\nu(1+\nu)}\,F_0\,\mathrm{Tr}\,(\eta\,\cdot)$$

$$+\;\frac{\eta\eta}{2\nu^2}\,F_0\,\mathrm{Tr}(\eta^+\eta^+\,\cdot)+\frac{\eta^+\eta^+}{2(1+\nu)^2}\,F_0\,\mathrm{Tr}\,(\eta\eta\,\cdot)\quad . \tag{A.12}$$

This object, however fails to fulfill eqs. (A.1) and (A.2). In fact

$$\bar{\mathbb{P}}F=\frac{\eta^+\eta-\nu}{1+\nu}\,F_0\quad . \tag{A.13}$$

The projector $\mathbb{P}$ is however imediately obtained by adding to $\bar{\mathbb{P}}$ terms which guarantee the validity of eq. (A.1), i.e.

$$\mathbb{P}\cdot\;=\;\bar{\mathbb{P}}\cdot-\frac{\eta^+\eta-\nu}{1+\nu}\,F_0\,\mathrm{Tr}(\cdot)+F_0\,\mathrm{Tr}(\cdot)\quad . \tag{A.14}$$

The full expression for the projector is thus

$$\mathbb{P}\cdot\;=\;\left\{\left[1-\frac{\eta^+\eta-\nu}{1+\nu}\right]\mathrm{Tr}(\cdot)+\frac{\eta^+\eta-\nu}{\nu(1+\nu)}\mathrm{Tr}(\eta^+\eta\,\cdot)+\frac{\eta}{\nu}\,\mathrm{Tr}(\nu^+\,\cdot)+\frac{\eta^+}{1+\nu}\,\mathrm{Tr}(\eta\,\cdot)\right.$$

$$\left.+\;\frac{\eta\eta}{2\nu^2}\,\mathrm{Tr}(\eta\eta^+\,\cdot)+\frac{\eta^+\eta^+}{2(1+\nu)^2}\,\mathrm{Tr}(\eta\eta\,\cdot)\right\}F_0\quad . \tag{A.15}$$

The construction of $\mathbb{P}^+$ follows immediately from the definition of scalar product:

$$(y\,,\mathbb{P}x)\;=\;\mathrm{Tr}[y^+(\mathbb{P}x)]\;=\;\mathrm{Tr}\left[y^+\left(1-\frac{\eta^+\eta-\nu}{1+\nu}\right)F_0\right]\mathrm{Tr}(x)$$

$$+\;\mathrm{Tr}\left[y^+\,\frac{\eta^+\eta-\nu}{\nu(1+\nu)}\,F_0\right]\mathrm{Tr}(\eta^+\eta x)+\mathrm{Tr}\left[y^+\,\frac{\eta}{\nu}\,F_0\right]\mathrm{Tr}(\eta^+x)$$

$$+\;\mathrm{Tr}\left[y^+\,\frac{\eta^+}{1+\nu}\,F_0\right]\mathrm{Tr}(\eta x)+\mathrm{Tr}\left[y^+\,\frac{\eta\eta}{2\nu^2}\,F_0\right]\mathrm{Tr}(\eta^+\eta^+x)$$

$$+\;\mathrm{Tr}\left[y^+\,\frac{\eta^+\eta^+}{2(1+\nu)^2}\,F_0\right]\mathrm{Tr}(\eta\eta x)\;=\;\mathrm{Tr}[(P^+\,y)^+\,x]\quad . \tag{A.16}$$

From this, one obtains immediately

$$\mathbb{P}^+\cdot\;=\;\mathrm{Tr}\left[F_0\left(1-\frac{\eta^+\eta-\nu}{1+\nu}\right)\cdot\right]+\frac{\eta^+\eta}{\nu(1+\nu)}\,\mathrm{Tr}[F_0(\eta^+\eta-\nu)\cdot]$$

$$+\;\frac{\eta}{\nu}\,\mathrm{Tr}[F_0\,\eta^+\,\cdot]+\frac{\eta^+}{1+\nu}\,\mathrm{Tr}[F_0\,\eta\,\cdot]+\frac{\eta\eta}{2\nu^2}\,\mathrm{Tr}[F_0\,\eta^+\eta^+\,\cdot]$$

$$+\;\frac{\eta^+\eta^+}{2(1+\nu)^2}\,\mathrm{Tr}[F_0\,\eta\eta\,\cdot]\quad . \tag{A.17}$$

Appendix B: Numerical Treatment of Memory Integrals

In order to obtain numerical solutions for the equations of motion, we need to evaluate memory integral of the type

$$I(t)\;=\;\int_0^t dt'\left(\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}}\,\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}}\,\frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right)_{t'}^{1/2}$$

$$\times\;\left(1+\sum_i\nu_{k_i}+\sum_{i<j}\nu_{k_i}\,\nu_{k_j}\right)_{t'}\langle\phi\rangle_{t'}\,\sin\left[\phi_{k_1}(t,t')+\phi_{k_2}(t,t')+\phi_{k_3}(t,t')\right]\quad . \tag{B.1}$$

Using the phase equation (5.6) explicitly this appears as

$$I(t)\;=\;-\sin\left\{\int_0^t dt'\,\left[f_{k_1}(t')+f_{k_2}(t')+f_{k_3}(t')\right]\right\}\int_0^t dt'$$

$$\times\;\left(\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}}\,\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}}\,\frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right)_{t'}^{1/2}\left(1+\sum_i^3\nu_{k_i}+\sum_{i<j}^3\nu_{k_i}\,\nu_{k_j}\right)_{t'}\langle\phi\rangle_{t'}$$

$$\times\;\cos\int_0^{t'}dt''\,\left[f_{k_1}(t'')+f_{k_2}(t'')+f_{k_3}(t'')\right]$$

$$+\;\cos\left\{\int_0^t dt'\,\left[f_{k_1}(t')+f_{k_2}(t')+f_{k_3}(t')\right]\right\}\int_0^t dt'$$

$$\times \left(\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}}\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}}\frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right)^{1/2}_{t'} \left(1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right)_{t'} \langle\phi\rangle_{t'}$$

$$\times \ \sin\int_0^{t'} dt'' \, [f_{k_1}(t'')+f_{k_2}(t'')+f_{k_3}(t'')] \quad . \tag{B.2}$$

In order to evaluate this a useful trick is to write a differential equation for the integral appearing in (B.2). Thus

$$\dot{I}_f = f_{k_1}(t)+f_{k_2}(t)+f_{k_3}(t) \tag{B.3}$$

$$\dot{I}_c = \left(\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}}\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}}\frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right)^{1/2} \times \left(1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right)\langle\phi\rangle\cos(I_f) \tag{B.4}$$

$$\dot{I}_s = \left(\frac{1}{\sqrt{k_1^2+\mu_{k_1}^2}}\frac{1}{\sqrt{k_2^2+\mu_{k_2}^2}}\frac{1}{\sqrt{k_3^2+\mu_{k_3}^2}}\right)^{1/2} \times \left(1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right)\langle\phi\rangle\sin(I_f) \tag{B.5}$$

so that

$$I(t) = -\sin(I_f)\cdot I_c + \cos(I_f)\cdot I_s \quad . \tag{B.6}$$

The differential equations (B.3), (B.4), (B.5) can be integrated easily by standard numerical methods together with the remaining dynamical equations.

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FIGURE CAPTIONS

Fig. 1: Time evolution of the expectation value of the field operator $\langle\phi\rangle$ (1A), mean-occupation number ν_k (1B) and dynamical effective mass μ_k (1C). Full line: collisional approximation; dashed line: mean-field approximation. See text for parameter values.

Fig. 2: Corresponding results to Fig. 1 for the case of $0+1$ dimensions. Parameter values: $m=\mu=1.1914879\,(m_B^2=1)$ and $g=2$; q is the mean position, μ_t is the dynamical effective mass and ν_t is the occupation number. Full line: exact solution; dashed line: collisional approximation; dotted line: mean-field approximation.

Fig. 3: Time evolution of the dynamical effective mass μ_k (Figs. 3A and 3B), mean-occupation number ν_k (Figs. 3C and 3D) and root-mean-square field (Fig. 3E) in broken symmetry potential with $m=0.05$ and $g=0.155$. Conventions are the same as in Fig. 1.

Fig. 4: Corresponding results to Fig. 1 for the case of $0+1$ dimensions. Parameter values: $m=0.05$ and $g=0,004$. Notations and conventions are as in Fig. 2.

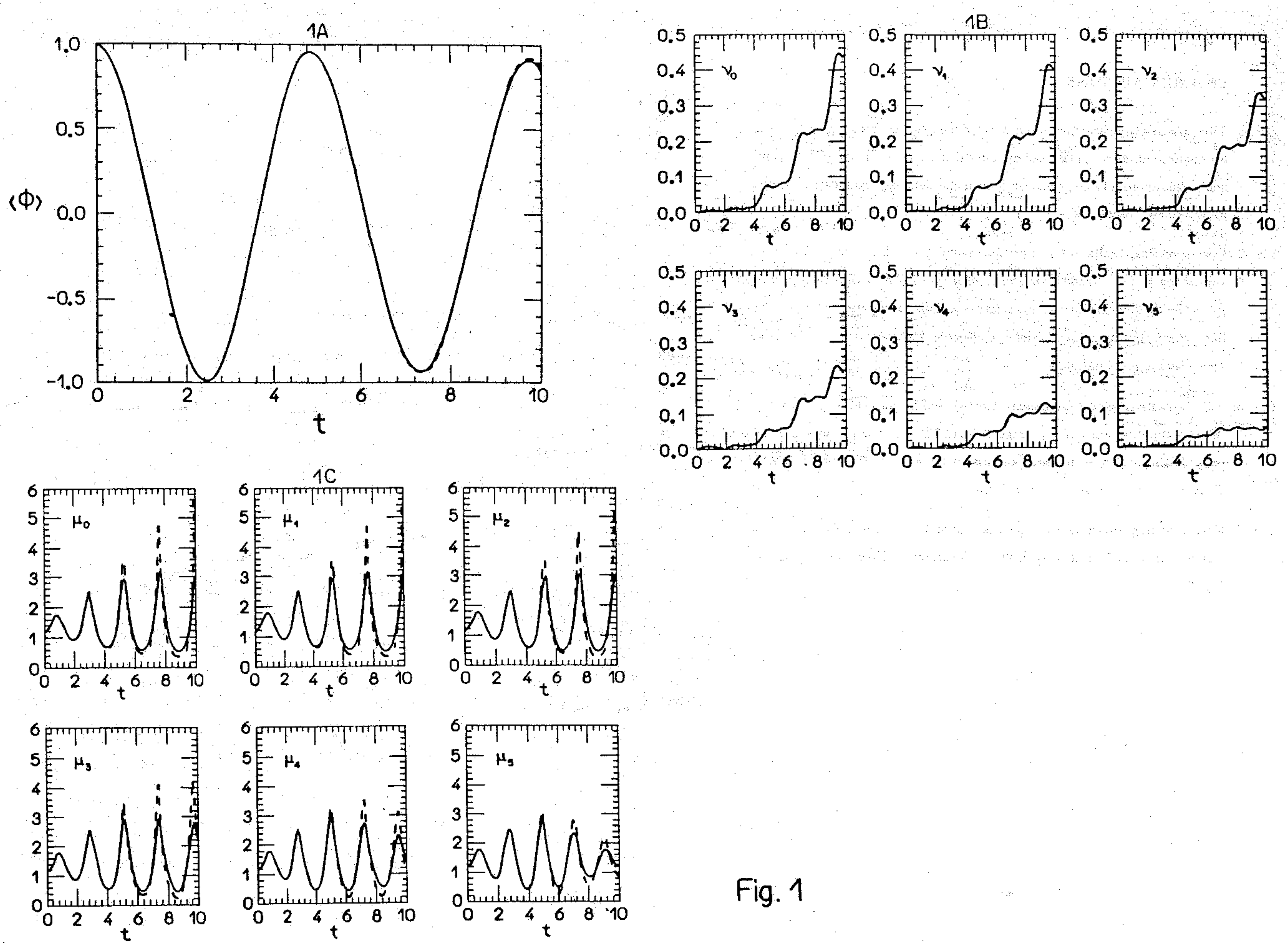


Fig. 1

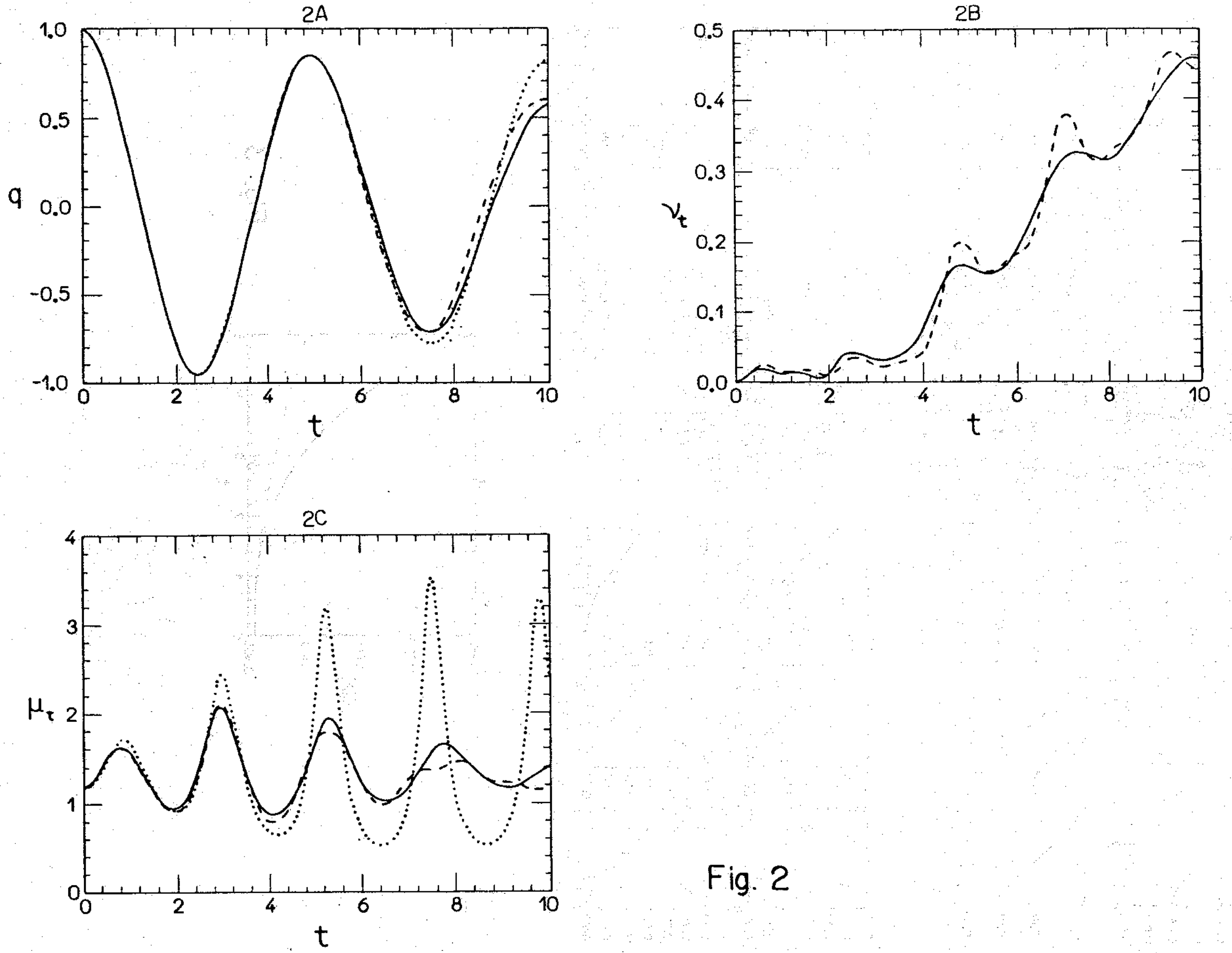

2A
1.0
0.5
q
0.0
-0.5
-1.0
0
2
4
6
8
10
t
2B
0.5
0.4
0.3
ν_t
0.2
0.1
0.0
0
2
4
6
8
10
t
2C
4
3
μ_τ
2
1
0
0
2
4
6
8
10
t

Fig. 2

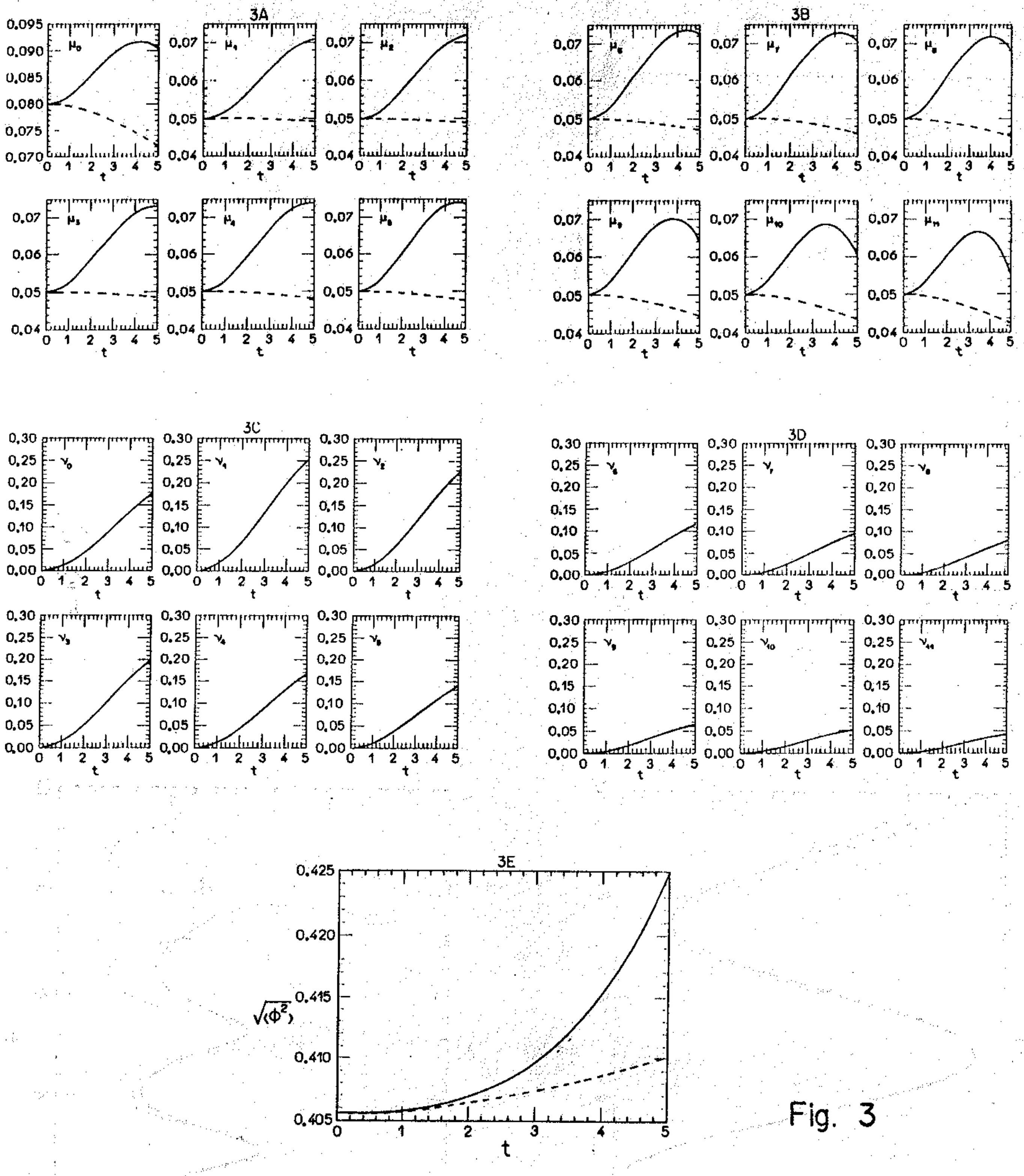
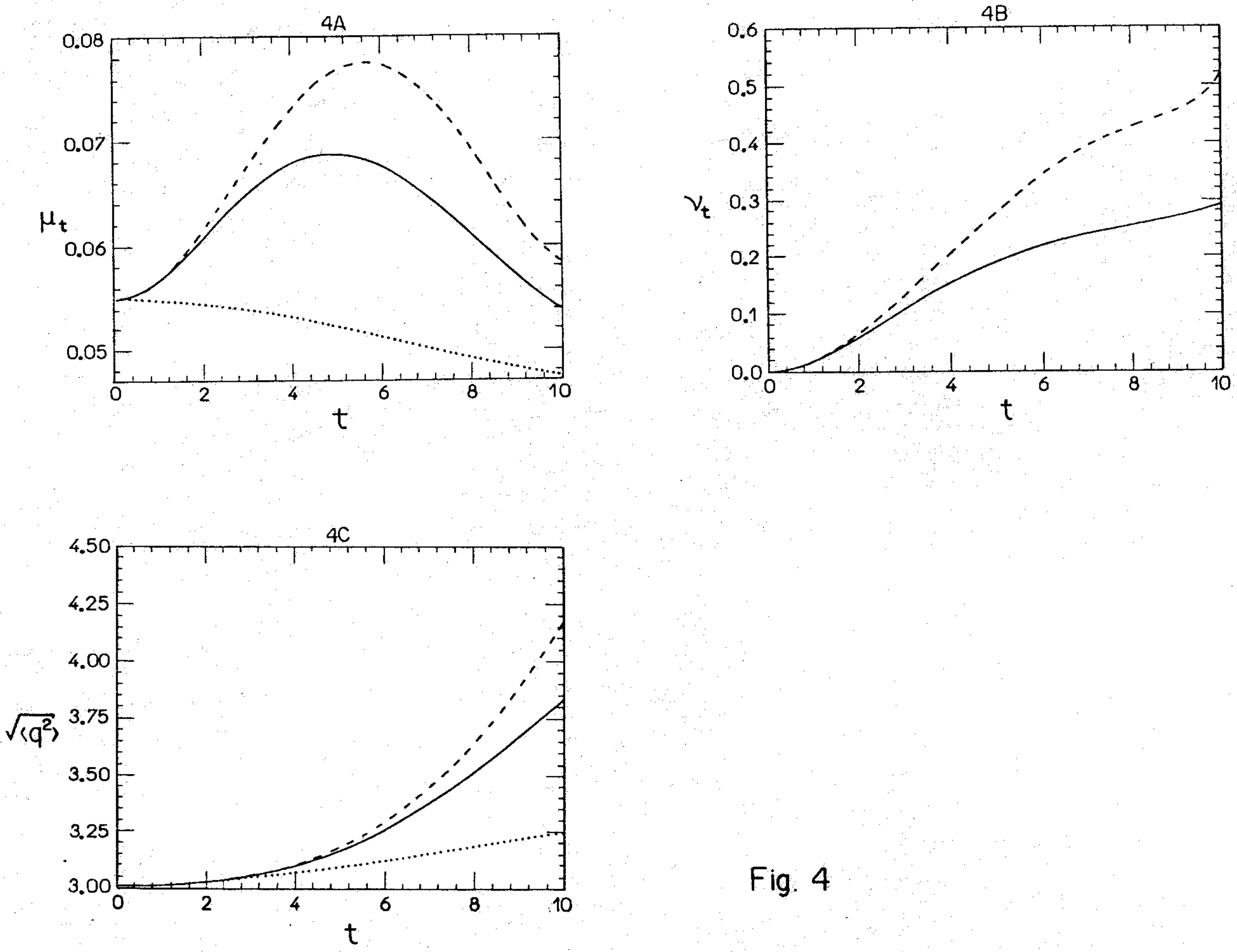

3A
3B
3C
3D
3E
μ_0
μ_1
μ_2
μ_3
μ_4
μ_5
μ_6
μ_7
μ_8
μ_9
μ_{10}
μ_{11}
ν_0
ν_1
ν_2
ν_3
ν_4
ν_5
ν_6
ν_7
ν_8
ν_9
ν_{10}
ν_{11}
t
$\sqrt{\langle\Phi^2\rangle}$

Fig. 3



4A
0.08
0.07
0.06
0.05
μt
0
2
4
6
8
10
t
4B
0.6
0.5
0.4
0.3
0.2
0.1
0.0
νt
0
2
4
6
8
10
t
4C
4.50
4.25
4.00
3.75
3.50
3.25
3.00
√⟨q²⟩
0
2
4
6
8
10
t

Fig. 4