UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 - SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-935

CORRELATION FUNCTIONS IN SUPER LIOUVILLE THEORY

E. Abdalla and K. Harada
Instituto de Física, Universidade de São Paulo

M.C.B. Abdalla and D. Dalmazi
Instituto de Física Teórica, UNESP
Rua Pamplona 145, CEP 01405 São Paulo, Brazil

Agosto/1991

Correlation functions in super Liouville theory

E. Abdalla[1], M.C.B. Abdalla[2],

D. Dalmazi[2], Koji Harada[1(a)]

[1] *Instituto de Física, Univ. São Paulo, CP 20516, São Paulo, Brazil*

[2] *Instituto de Física Teórica, UNESP, Rua Pamplona 145, CEP 01405, São Paulo, Brazil*

We calculate three- and four-point functions in super Liouville theory coupled to super Coulomb gas on world sheets with spherical topology. We first integrate over the zero mode and assume that a parameter takes an integer value. After calculating the amplitudes, we formally continue the parameter to an arbitrary real number. Remarkably the result is completely parallel to the bosonic case, the amplitudes being of the same form as those of the bosonic case.

Typeset Using *REVTEX*

The matrix model definition of 2D-gravity has been proving to be very powerful in calculating correlation functions[1]. It seems, however, difficult to generalize the results to supersymmetric theories.

On the other hand, in the continuum approach (Liouville theory)[2–5] it is difficult to calculate correlation functions, while its supersymmetric generalization (super Liouville theory)[6–8] is well known. Recently, however, several authors[9–13] have exactly calculated correlation functions in the continuum approach to conformal matter fields coupled to 2D-gravity. (See also Ref. [14]). They have used a technique based on the integration over the Liouville zero mode, and their results agree with those obtained earlier in the discrete approach (matrix models). Since a supersymmetric generalization of the matrix models has not yet appeared it is very urgent to extend the continuum method[15] to the supersymmetric case, i.e., superconformal matter fields coupled to 2D-supergravity.

The aim of this Letter is to calculate the three- and four-point functions in super Liouville theory coupled to superconformal matter with the central charge $\hat{c} < 1$, represented as super Coulomb gas[16]. Our approach is close to that of Di Francesco and Kutasov[10]. The result is remarkable and is very parallel to the bosonic case; it amounts to a redefinition of the cosmological constant and of the primary superfields, resulting the same amplitudes as those of the bosonic theory.

The relevant framework has been given by Distler, Hlousek and Kawai[8]. With a translation invariant measure, the total action for super Liouville theory coupled to superconformal matter is given by

$$S = S_{SL} + S_M \quad ,$$

$$S_{SL} = \frac{1}{4\pi}\int d^2\mathbf{z}\hat{E}\left(\frac{1}{2}\hat{D}_\alpha\Phi_{SL}\hat{D}^\alpha\Phi_{SL} - Q\hat{Y}\Phi_{SL} - 4i\mu e^{\alpha_+\Phi_{SL}}\right) \quad , \tag{1}$$

$$S_M = \frac{1}{4\pi}\int d^2\mathbf{z}\hat{E}(\frac{1}{2}\hat{D}_\alpha\Phi_M\hat{D}^\alpha\Phi_M + 2i\alpha_0\hat{Y}\Phi_M) \quad , \tag{2}$$

where Φ_{SL}, Φ_M, are super Liouville and matter superfields respectively. (For the superfield notation, we refer the reader to Refs. [8, 17]). The matter sector has the central charge $\hat{c}_m = 1 - 8\alpha_0^2$. By using this, the parameters Q and $\alpha_\pm$ are given by

$$Q = 2\sqrt{1+\alpha_0^2} \quad , \quad \alpha_\pm = -\frac{Q}{2} \pm \frac{1}{2}\sqrt{Q^2-4} = -\frac{Q}{2} \pm |\alpha_0| \quad . \tag{3}$$

The (gravitationally dressed) primary superfield $\bar{\Psi}_{NS}$ is given by

$$\bar{\Psi}_{NS}(\mathbf{z}, k) = d^2\mathbf{z}\hat{E}e^{ik\Phi_M(\mathbf{z})}e^{\beta(k)\Phi_{SL}(\mathbf{z})} \tag{4}$$

with

$$\beta(k) = -\frac{1}{2}Q + |k - \alpha_0| \quad . \tag{5}$$

Screening charges in the matter sector are of the form $d^2\mathbf{z}e^{id_\pm\Phi_M(\mathbf{z})}$, where $d_\pm$ are the two solutions of $\frac{1}{2}d(d-2\alpha_0) = \frac{1}{2}$. In this Letter, however, we will concentrate on the case without screening charges. The case with screening charges, $N(\geq 4)$- point functions and the inclusion of the Ramond sector will be discussed elsewhere.

We shall calculate three-point functions of the primary field $\bar{\Psi}_{NS}$ on world sheets with spherical topology (without screening charges), that is,

$$\left\langle \prod_{i=1}^{3} \int \bar{\Psi}_{NS}(\mathbf{z}_i, k_i) \right\rangle \equiv \int [\mathcal{D}_{\hat{E}}\Phi_{SL}][\mathcal{D}_{\hat{E}}\Phi_M] \prod_{i=1}^{3} \bar{\Psi}_{NS}(\mathbf{z}_i, k_i) e^{-S} \quad . \tag{6}$$

Our first step is to integrate over the zero modes.

$$\left\langle \prod_{i=1}^{3} \int \bar{\Psi}_{NS}(\mathbf{z}_i, k_i) \right\rangle \equiv 2\pi\delta\left(\sum_{i=1}^{3} k_i - 2\alpha_0\right) \mathcal{A}(k_1, k_2, k_3),$$
$$\mathcal{A}(k_1,k_2,k_3) = \Gamma(-s)(\frac{-\pi}{2})^3(\frac{i\mu}{\pi})^s \left\langle \int \prod_{i=1}^{3} d^2\tilde{\mathbf{z}}_i e^{ik_i\Phi_M(\tilde{\mathbf{z}}_i)} e^{\beta_i\Phi_{SL}(\tilde{\mathbf{z}}_i)} \left(\int d^2\mathbf{z} e^{\alpha_+\Phi_{SL}(\mathbf{z})}\right)^s \right\rangle_0, \tag{7}$$

where $\langle\cdots\rangle_0$ denotes the expectation value evaluated in the free theory ($\mu = 0$) and we have absorbed the factor $[\alpha_+(-\pi/2)^3]^{-1}$ into the normalization of the path integral. The parameter s is defined as

$$s = -\frac{1}{\alpha_+}\left[Q + \sum_{i=1}^{3}\beta_i\right] \quad . \tag{8}$$

In general, s can take any real value and there is no obvious way of calculating the path-integral. However, if we assume that s is a non-negative integer, as in Ref. [8–13], this is just a free-field correlator. Under this assumption, we evaluate the path-integral, and formally continue s to non-integer values. For s non-negative, we have

$$\begin{aligned}
\mathcal{A}(k_1,k_2,k_3) &= \Gamma(-s)(\frac{-\pi}{2})^3\left(\frac{i\mu}{\pi}\right)^s \\
&\times \int \prod_{i=1}^{3} d^2\tilde{\mathbf{z}}_i \prod_{i=1}^{s} d^2\mathbf{z}_i \prod_{i<j}^{3} |\tilde{\mathbf{z}}_{ij}|^{2k_ik_j-2\beta_i\beta_j} \prod_{i=1}^{3}\prod_{j=1}^{s} |\tilde{z}_i - z_j - \tilde{\theta}_i\theta_j|^{-2\alpha_i\beta_i} \prod_{i<j}^{s} |\mathbf{z}_{ij}|^{-2\alpha_+^2} \\
&= \Gamma(-s)(\frac{-\pi}{2})^3\left(\frac{i\mu}{\pi}\right)^s \int \prod_{i=1}^{s} d^2z_i d^2\tilde{\theta} \prod_{i=1}^{s} |z_i + \tilde{\theta}\theta_i|^{-2\alpha_+\beta_1} \prod_{1=1}^{s} |1 - z_i|^{-2\alpha_+\beta_2} \prod_{i<j}^{s} |\mathbf{z}_{ij}|^{-2\alpha_+^2}
\end{aligned} \tag{9}$$

We have divided by the $\widehat{SL_2}$ volume by setting $\tilde{z}_1 = 0, \tilde{z}_2 = 1, \tilde{z}_3 = \infty, \tilde{\theta}_2 = \tilde{\theta}_3 = 0$ and $\tilde{\theta}_1 \equiv \tilde{\theta}$. The integral is the supersymmetric generalization of (B.9) of Ref.[18]. Alternatively, it is expressed in the following way by using the components of $\Phi_{SL} = \phi + \theta\psi + \bar{\theta}\bar{\psi}$.

$$\begin{aligned}
&\mathcal{A}(k_1,k_2,k_3) \\
&= \Gamma(-s)(\frac{-\pi}{2})^3\left(\frac{i\alpha_+^2\mu}{\pi}\right)^s \beta_1^2 \int \prod_{i=1}^{s} d^2z_i \prod_{i=1}^{s} |z_i|^{-2\alpha_+\beta_1} |1 - z_i|^{-2\alpha_+\beta_2} \prod_{i<j}^{s} |z_i - z_j|^{-2\alpha_+^2} \\
&\qquad \left\langle \overline{\psi}\psi(0)\overline{\psi}\psi(z_1)\cdots\overline{\psi}\psi(z_s)\right\rangle_0
\end{aligned} \tag{10}$$

We first observe that this is non-vanishing only for s odd; we thus write $s \equiv 2m + 1$. One may evaluate $\langle\overline{\psi}\cdots\overline{\psi}\rangle_0$ and $\langle\psi\cdots\psi\rangle_0$ independently. Since the rest of the integrand is symmetric, one may write the result in a simple form by relabelling coordinates:

$$\mathcal{A}(k_1,k_2,k_3) = \Gamma(-s)(\frac{-\pi}{2})^3\frac{1}{\alpha_+^2}\left(\frac{i\alpha_+^2\mu}{\pi}\right)^s \alpha^2(-1)^{m+1}(2m+1)!!$$

$$\times \int \prod_{i=1}^{2m+1} d^2 z_i \prod_{i=1}^{2m+1} |z_i|^{2\alpha}|1-z_i|^{2\beta} \prod_{i<j}^{2m+1} |z_i - z_j|^{4\rho} \prod_{i=1}^{m} |z_{2i-1} - z_{2i}|^{-2} |z_{2m+1}|^{-2}$$
$$= -i(\frac{-\pi}{2})^3 \Gamma(-s)\Gamma(s+1)\frac{1}{\alpha_+^2}\left(\frac{\alpha_+^2\mu}{\pi}\right)^s I^m(\alpha,\beta;\rho) \tag{11}$$

where

$$\begin{aligned} &I^m(\alpha,\beta;\rho)\\ &= \frac{1}{2^m m!}\alpha^2 \int d^2 w \prod_{i=1}^{m} d^2\zeta_i d^2\eta_i |w|^{2\alpha-2}|1-w|^{2\beta} \prod_{i=1}^{m} |w-\zeta_i|^{4\rho}|w-\eta_i|^{4\rho}\\ &\times \prod_{i=1}^{m} |\zeta_i|^{2\alpha}|\eta_i|^{2\alpha}|1-\zeta_i|^{2\beta}|1-\eta_i|^{2\beta} \prod_{i,j}^{m} |\zeta_i - \eta_j|^{4\rho} \prod_{i<j}^{m} |\zeta_i-\zeta_j|^{4\rho}|\eta_i-\eta_j|^{4\rho} \prod_{i=1}^{m} |\zeta_i - \eta_i|^{-2} \end{aligned} \tag{12}$$

and $\alpha = -\alpha_+\beta_1$, $\beta = -\alpha_+\beta_2$, $\rho = -\frac{1}{2}\alpha_+^2$.

We now have to calculate $I^m(\alpha,\beta;\rho)$. First of all, we assume that $I^m(\alpha,\beta;\rho)$ is symmetric in α and β: $I^m(\alpha,\beta;\rho) = I^m(\beta,\alpha;\rho)$. It is easy to check that when $m=0$. The large-α and large-β behaviors are consistent with this assumption and it is physically natural because the amplitude should be symmetric under the exchange of two external momenta. Under this assumption, $I^m(\alpha,\beta;\rho)$ exhibits the following symmetry

$$I^m(\alpha,\beta;\rho) = I^m(-1-\alpha-\beta-m\rho,\beta;\rho) \tag{13}$$

Thus we may write $I^m(\alpha,\beta;\rho)$ in the following way

$$\begin{aligned} &I^m(\alpha,\beta;\rho)\\ &= C_m(\alpha,\beta;\rho)\prod_{i=0}^{m}\Delta(1+\alpha+2i\rho)\Delta(1+\beta+2i\rho)\Delta(-\alpha-\beta+(2i-4m)\rho)\\ &\times \prod_{i=1}^{m}\Delta(\frac{1}{2}+\alpha+(2i-1)\rho)\Delta(\frac{1}{2}+\beta+(2i-1)\rho)\Delta(-\frac{1}{2}-\alpha-\beta+(2i-4m-1)\rho) \end{aligned} \tag{14}$$

where $C_m(\alpha,\beta;\rho)$ has the same symmetries as $I^m(\alpha,\beta;\rho)$, and where $\Delta(x) \equiv \Gamma(x)/\Gamma(1-x)$. By looking at the large-α behavior:

$$I^m(\alpha,\beta;\rho) \sim \alpha^{-2m-2(2m+1)\beta-4\rho m(2m+1)} \quad , \tag{15}$$

one can confirm that $C_m(\alpha,\beta;\rho)$ is, as a function of α, bounded as $|\alpha| \to \infty$ and analytic. This means that $C_m(\alpha,\beta;\rho)$ is independent of α, and by symmetry, of β as well; $C_m = C_m(\rho)$.

It is hard to calculate $C_m(\rho)$. For this purpose, it is useful to consider the simpler integral:

$$\begin{aligned} J^m(\alpha,\beta;\gamma;\rho) = \int \prod_{i=1}^{m} d^2\zeta_i d^2\eta_i \prod_{i=1}^{m} |\zeta_i|^{2\alpha}|\eta_i|^{2\alpha}|1-\zeta_i|^{2\beta}|1-\eta_i|^{2\beta} \prod_{i,j}^{m} |\zeta_i-\eta_j|^{4\rho}\\ \times \prod_{i<j}^{m} |\zeta_i-\zeta_j|^{4\rho}|\eta_i-\eta_j|^{4\rho} \prod_{i=1}^{m} |\zeta_i-\eta_i|^{4\gamma} \quad . \end{aligned} \tag{16}$$

By using similar arguments, one may obtain

$$\begin{aligned} &J^m(\alpha,\beta;\gamma;\rho)\\ &= \tilde{C}_m(\gamma;\rho)\prod_{i=0}^{m-1}\Delta(1+\alpha+2i\rho)\Delta(1+\beta+2i\rho)\Delta(-1-\alpha-\beta-2\gamma+(2i-4m+2)\rho)\\ &\times \prod_{i=1}^{m}\Delta(1+\alpha+\gamma+(2i-1)\rho)\Delta(1+\beta+\gamma+(2i-1)\rho)\Delta(-1-\alpha-\beta-\gamma+(2i-4m+2)\rho) \end{aligned} \tag{17}$$

Again, it is very difficult to calculate $\tilde{C}_m(\gamma;\rho)$. Unfortunately we could not get it in a rigorous way. A series of trials and errors, however, led us to the following form;

$$\tilde{C}_m(\gamma;\rho) = \frac{\pi^{2m}}{2^m} m! \left[\Delta\left(-(\gamma+\rho)\right)\right]^{2m} \prod_{i=1}^{m} \Delta\left(1+2(\gamma+i\rho)\right)\Delta\left(1+\gamma+(2i-1)\rho\right) \quad . \tag{18}$$

This is consistent with

$$\tilde{C}_1(\gamma;\rho) = \frac{\pi^2}{2}\left[\Delta\left(-(\gamma+\rho)\right)\right]^2 \Delta\left(1+\gamma+\rho\right)\Delta\left(1+2(\gamma+\rho)\right) \quad , \tag{19}$$

and the two other (calculable) cases $\rho = 0$ and $\gamma = 0$ (up to numerical coefficients, i.e., symmetry factors). It is very difficult to get anything else consistent with these constraints. And *a posteriori* it seems to be correct since it gives a physically reasonable result. Let us assume that (18) is correct and see its consequences.

The two integrals are related by

$$I^m(\epsilon,\beta;\rho) = -\frac{\pi}{2^m m!}\Delta(1+\epsilon)\Delta(1+\beta)\Delta(-\epsilon-\beta)J^m(2\rho,\beta;-1/2;\rho) . \quad (20)$$

Therefore $C_m(\rho) = -(\pi/2^m m!)\tilde{C}_m(-1/2,\rho)\Delta(\frac{1}{2}-\rho)\Delta(\frac{1}{2}+(2m+1)\rho)$. If we substitute (18) we get

$$C_m(\rho) = -\frac{\pi^{2m+1}}{2^{2m}}\left[\Delta\left(\frac{1}{2}-\rho\right)\right]^{2m+1}\prod_{i=1}^{m}\Delta(2i\rho)\prod_{i=0}^{m}\Delta\left(\frac{1}{2}+(2i+1)\rho\right) \quad . \quad (21)$$

Now we are ready to write down the amplitude. Without loss of generality, we can choose $k_1, k_3 \geq \alpha_0, k_2 \leq \alpha_0$. By using (5), (8) and $\sum_{i=1}^{3} k_i = 2\alpha_0$, one gets

$$\beta = \begin{cases} 2\rho(1-s) \ (\alpha_0 > 0) \\ -1-2\rho s \ (\alpha_0 < 0). \end{cases}$$

It is easily seen that, for $\alpha_0 > 0$, $\mathcal{A} = 0$ identically, as in the bosonic theory. For $\alpha_0 < 0$, there are many cancellations (independent of (18)), leading to

$$\begin{aligned} &I^m(\alpha,\beta,\rho) \\ &= C_m(\rho)\prod_{i=0}^{m}\Delta\left(\frac{1}{2}-(2i+1)\rho\right)\prod_{i=1}^{m}\Delta(-2i\rho)\Delta(1+\alpha+2m\rho)\Delta\left(\frac{1}{2}-\alpha+\rho\right) \\ &= (-1)^{m+1}\frac{\pi^{2m+1}}{(m!)^2}\left[\Delta\left(\frac{1}{2}-\rho\right)\right]^{2m+1}(4\rho)^{-2m}\Delta(1+\alpha+2m\rho)\Delta\left(\frac{1}{2}-\alpha+\rho\right) \quad (22) \end{aligned}$$

We finally obtain the three-point function.

$$\begin{aligned} \mathcal{A}(k_1,k_2,k_3) &= (\frac{-i\pi}{2})^3\left[\frac{\mu}{2}\Delta\left(\frac{1}{2}-\rho\right)\right]^s\Delta\left(\frac{1}{2}-\frac{s}{2}\right)\Delta(1+\alpha-2m\rho)\Delta\left(\frac{1}{2}-\alpha+\rho\right) \\ &= \left[\frac{\mu}{2}\Delta\left(\frac{1}{2}-\rho\right)\right]^s\prod_{j=1}^{3}(-\frac{i\pi}{2})\Delta\left(\frac{1}{2}[1+\beta_j^2-k_j^2]\right) \quad (23) \end{aligned}$$

By redefining the cosmological constant and the primary superfield $\bar{\Psi}_{NS}$ as

$$\mu \to \frac{2}{\Delta\left(\frac{1}{2}-\rho\right)}\mu \quad , \quad \bar{\Psi}_{NS}(k_j) \to \frac{1}{(-\frac{i}{2}\pi)\Delta\left(\frac{1}{2}[1+\beta_j^2-k_j^2]\right)}\bar{\Psi}_{NS}(k_j) \quad , \quad (24)$$

we get our main result

$$\mathcal{A}(k_1,k_2,k_3) = \mu^s \quad . \quad (25)$$

Remarkably, this amplitude is of the same form as the bosonic one[10].

It is natural to expect that this feature continues to be true for $N(\geq 4)$- point functions. Work in this direction is in progress.

In fact, for $k_1, k_2, k_3 \geq \alpha_0$, $k_4 \leq \alpha_0 < 0$ (and without screening charges), the four point function turns out to be

$$\mathcal{A}(k_1,k_2,k_3,k_4) = (s+1)\mu^s \, , \quad (26)$$

with the same redefinition of the cosmological constant and the primary superfields. In order to get the amplitude for general k_i, one may argue, as in Ref. [10], that non-analyticity comes entirely from massless intermediate states and one may calculate the amplitude by using the analyticity of the one particle irreducible (1PI) correlators. After setting $\mu = 1$, we obtain the four-point function for all k_i:

$$\mathcal{A}(k_1,k_2,k_3,k_4) = \alpha_-[|k_1+k_2-\alpha_0|+|k_1+k_3-\alpha_0|+|k_1+k_4-\alpha_0|]+\mathcal{A}_{1PI}, \quad (27)$$

with $\mathcal{A}_{1PI} = -\frac{1}{2}(1+\alpha_-^2)$. Compare with Eq. (37) in Ref. [10]. The analogy to the bosonic case is obvious. A detailed account will appear elsewhere.

In a recent paper Alvarez-Gaumé and Mañes[19] considered a general class of supermatrix models and found that all those models are strictly equivalent to theories based on ordinary matrices. Although it seems that they looked at completely different things, our result might suggest that the discretization of 2D supergravity should give the same answers as those of the ordinary one.

In conclusion, we calculated the three- and four-point functions of super Liouville theory coupled to super Coulomb gas (without screening charges) on a sphere and found that they are essentially the same as those of the usual Liouville theory, obtained in Ref. [11]. As a by-product we get the supersymmetric generalization of (B.9) formula of Ref. [18],

$$
\begin{aligned}
&\int \prod_{i=1}^{m} d^2 z_i d^2\bar{\theta} \prod_{i=1}^{m} |z_i + \bar{\theta}\theta_i|^{2\alpha} \prod_{i=1}^{m} |1 - z_i|^{2\beta} \prod_{i<j}^{m} |z_{ij}|^{4\rho} = \\
&(-1)^m \pi^{2m+1} (2m+1)! \rho^{2m} \Delta\left(\frac{1}{2} - \rho\right)^{2m+1} \prod_{i=1}^{m} \Delta(2i\rho) \prod_{i=0}^{m} \Delta\left(\frac{1}{2} + (2i+1)\rho\right) \\
&\times \prod_{i=0}^{m} \Delta(1 + \alpha + 2i\rho)\Delta(1 + \beta + 2i\rho)\Delta(-\alpha - \beta + (2i - 4m)\rho) \\
&\times \prod_{i=1}^{m} \Delta\left(\frac{1}{2} + \alpha + (2i-1)\rho\right)\Delta\left(\frac{1}{2} + \beta + (2i-1)\rho\right)\Delta\left(-\frac{1}{2} - \alpha - \beta + (2i - 4m - 1)\rho\right) ,
\end{aligned}
\tag{28}
$$

which may be useful in further developments.

ACKNOWLEDGMENTS

K.H. would like to thank the members of Instituto de Física Teórica, UNESP, for their hospitality extended to him and FAPESP for the financial support under contract number 90/1799-9. He also acknowledges communications on 2D gravity with Y.Tanii and with N.Ishibashi. D.D thanks FAPESP for financial support under contract number 90/2246-3. Work of E.A. and M.C.B.A. is partially supported by CNPq.

(a) Address after October 1991, Department of Physics, Kyushu University, Fukuoka 812, Japan.

REFERENCES

[1] E. Brézin and V. A. Kazakov, Phys. Lett. **B236**,144 (1990); M. R. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990); D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990).

[2] A. M. Polyakov, Phys. Lett. **B103**, 207 (1981).

[3] T. L. Curtright and C. B. Thorn, Phys. Rev. Lett. **48**, 1309 (1982); E. Braaten, T. L. Curtright and C. B. Thorn, Phys. Lett. **B118**, 115 (1982); Ann. Phys. **147**, 365 (1983); E. Braaten, T. L. Curtright, G. Gandour and C. B. Thorn, Phys. Rev. Lett. **51**, 19 (1983); Ann. Phys. **153**, 147 (1984); J. L. Gervais and A. Neveu, Nucl. Phys. **B199**, 50 (1982); **B209**, 125 (1982); **B224**, 329 (1983); **B238**, 123,396 (1984); E. D'Hoker and R. Jackiw, Phys. Rev. **D26**, 3517 (1982); T. Yoneya, Phys. Lett. **B148**, 111 (1984).

[4] F. David, Mod. Phys. Lett. **A3**, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. **B321**, 509 (1989).

[5] N. Seiberg, Lecture at 1990 Yukawa Int. Sem. Common Trends in Math. and Quantum Field Theory, and Cargese meeting Random Surfaces, Quantum Gravity and Strings, May 27, June 2, 1990; J. Polchinski, Strings '90 Conference, College Station, TX, Mar 12-17, 1990, Nucl. Phys. **B357** 241 (1991).

[6] A. M. Polyakov Phys. Lett. **B103** , 211 (1981).

[7] J. F. Arvis, Nucl. Phys. **B212**, 151 (1983); **B218**, 309 (1983); O. Babelon, Nucl. Phys. **B258**, 680 (1985); Phys. Lett. **141B**, 353 (1984); T. Curtright and G. Ghandour, Phys. Lett. **B136**, 50 (1984).

[8] J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. **A5**, 391 (1990).

[9] M. Goulian and M. Li, Phys. Rev. Lett. **66**, 2051 (1991).

[10] P. Di Francesco and D. Kutasov, Phys. Lett. **B261**, 385 (1991).

[11] Vl. S. Dotsenko, preprint PAR-LPTHE 91-18 (1991).

[12] Y. Kitazawa, preprint HUTP-91/A013 (1991).

[13] N. Sakai and Y. Tanii, preprint TIT/HEP-169 (STUPP-91-116) (1991).

[14] A. Gupta, S. P. Trivedi and M. B. Wise, Nucl. Phys. **B340**, 475 (1990).

[15] (See also) M. Bershadsky and I. R. Klebanov, Phys. Rev. Lett. **65**, 3088 (1990); preprint HUTP-91/A002 (PUPT-1236) (1991); A. M. Polyakov, Mod. Phys. Lett. **A6**, 635 (1991).

[16] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, Phys. Lett. **B151**, 131 (1985); D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. **B151**, 37 (1985).

[17] E. Martinec, Phys. Rev. **D28**, 2604 (1983).

[18] Vl. S. Dotsenko and V. Fateev, Nucl. Phys. **B240**, 312 (1984); **B251**, 691 (1985).

[19] L. Alvarez-Gaumé and J. L. Mañez, Mod. Phys. Lett. **A6**, 2039 (1991).