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Abstract

We present an action for a spinning particle which is invariant by worldline reparametrization and worldline supersymmetry although it does not depend on the einbein. Upon quantization the theory describes topological field theories of the BF and Chern-Simons types. The classical theory can be coupled to a background gravitational field and therefore it is not a topological theory. A superspace formulation of the spinning topological particle is also given.

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One of the main problems in string theory is to find its second quantized version, that is, a field theory for strings and its fundamental symmetries. This showed in fact that even in the point particle case the transition from a first quantized theory to a second quantized one in not well understood. In the point particle case an action for a spinning particle with N extended supersymmetries was shown to give rise to a field theory with spin $\frac{N}{2}$ upon quantization [1]. The quantization by the Dirac method [1] or by path integral methods [2, 3] does not reveal the gauge structure for $N \geq 2$ since only field strengths show up in the quantized theory.

Even so it is very interesting by itself to find out gauge theories as a result of the quantization of spinning particles. This gave rise to a variant of the $N = 2$ model, by adding a Chern-Simons like term, and the quantization yielded the theory of antisymmetric tensors [4]. The path integral quantization of the theory was also performed [2, 5].

We now propose an action for a topological spinning particle which when quantized describes topological field theories [6]. The spinning particle is topological in the sense that its action does not depend on the einbein although it is reparametrization invariant in the same way that a topological field theory does not depend on the space-time metric tensor and the action is invariant under general coordinate transformations. Another relevant characteristic of this particle is that its Lagrangian is a pure BRST transformation.

The action is given by

$$S = \int_{\tau_1}^{\tau_2} d\tau (P_\mu \dot{X}^\mu - i\Pi_\mu \dot{\psi}^\mu + i\lambda P_\mu \psi^\mu) \tag{1}$$

where X^μ is the particle coordinate, ψ^μ the Grassmannian co-

ordinate, P_μ and Π_μ are the momenta canonically conjugated to X^μ and ψ^μ respectively and λ is the worldline gravitino field. The action (1) is in first order form and it is not quadratic in P_μ so that the worldline einbein is absent. Even so it is invariant by worldline reparametrizations

$$\begin{aligned} \delta X^\mu = \epsilon \dot{X}^\mu \quad &, \quad \delta\psi^\mu = \epsilon\dot{\psi}^\mu \\ \delta P_\mu = \epsilon \dot{P}_\mu \quad &, \quad \delta\Pi_\mu = \epsilon\dot{\Pi}_\mu \\ \delta\lambda = (\epsilon\lambda)^{\cdot} & \end{aligned} \tag{2}$$

if the parameter ϵ satisfies the boundary conditions $\epsilon(\tau_1) = \epsilon(\tau_2) = 0$. It is also invariant by worldline supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= -i\xi\psi^\mu \\ \delta\psi^\mu &= -\frac{1}{4e}\xi(\dot{X}^\mu + i\lambda\psi^\mu) \\ \delta P_\mu &= -\frac{i}{4e}\xi(\dot{\Pi}_\mu + \lambda P_\mu) \\ \delta e &= \frac{i}{2}\xi\lambda \\ \delta\lambda &= \dot{\xi} \end{aligned} \tag{3}$$

where e is the einbein. The supersymmetry parameter must also satisfy the boundary conditions $\xi(\tau_1) = \xi(\tau_2) = 0$. The supersymmetry transformations (3) close the algebra

$$[\delta_1, \delta_2] = \delta_3 + \delta_\epsilon \tag{4}$$

where δ_3 is a supersymmetry transformation with parameter $\xi_3 = \frac{i\xi_1\xi_2\lambda}{2e}$ and δ_ϵ is a reparametrization with parameter $\epsilon = -\frac{i\xi_1\xi_2}{2e}$.

Although being invariant by two local transformations the action (1) has only one (fermionic) constraint $Q = iP_\mu\psi^\mu$, which generates an abelian algebra. The local transformations generated by Q are

$$\begin{aligned} \delta X^\mu &= -i\alpha\psi^\mu \\ \delta\psi^\mu &= \delta P_\mu = 0 \\ \delta\Pi_\mu &= -\alpha P_\mu \\ \delta\lambda &= \dot{\alpha} \end{aligned} \tag{5}$$

and the action (1) is invariant under the transformations (5) without any boundary conditions on the Grassmannian parameter α. Since we have only one constraint the transformations (2) and (3) can not be independent and must be proportional do (5) up to field equations. In fact we find for the reparametrizations (2) that

$$\delta_\epsilon = \delta_\alpha + \epsilon\{\cdot, S\} \tag{6}$$

where δ_α is a transformation (5) with parameter $\alpha = \epsilon\lambda$ and $\{\cdot, S\}$ is the functional Poisson bracket with the action S. It is remarkable that even without the usual constraint $P^2 = 0$ we have reparametrization invariance and most remarkable it is being generated by a fermionic constraint. For the supersymmetry transformations (3) we have

$$\begin{aligned} \delta_\xi X^\mu &= \delta_\alpha X^\mu \\ \delta_\xi \Pi_\mu &= \delta_\alpha \Pi_\mu \\ \delta_\xi \psi^\mu &= \delta_\alpha \psi^\mu + \frac{\delta S}{\delta P_\mu}\Lambda \\ \delta_\xi P_\mu &= \delta_\alpha P_\mu + \frac{\delta S}{\delta \psi^\mu}\Lambda \end{aligned}$$

$$\delta_\xi\lambda = \delta_\alpha\lambda \tag{7}$$

where $\alpha = \xi$ and $\Lambda = -\frac{\xi}{4e}$. We have a situation similar to that of a topological field theory where the general coordinate transformation turns out to be a combination of a gauge transformation and field equations.

The constraint $Q = iP_\mu\psi^\mu$ can also be thought of as a BRST charge since it is fermionic and nilpotent. If we drop the parameter α in (5) we can reinterpret them as BRST transformations and the action (1) can be rewritten as a pure BRST transformation

$$S = \int_{\tau_1}^{\tau_2} d\tau\,\delta(-\Pi_\mu\dot{X}^\mu + i\lambda\Pi_\mu\psi^\mu) \tag{8}$$

which is also characteristic of topological field theories.

The Dirac quantization can be performed be realizing P_μ and Π_μ as differential operators $\hat{P}_\mu = -i\partial_\mu$ and $\hat{\Pi}_\mu = \frac{\partial}{\partial\psi^\mu}$, and by imposing the constraint $Q = 0$ on the physical states. The wave function Ψ will depend on the coordinates $\hat{X}^\mu = x^\mu$ and $\hat{\psi}^\mu = \psi^\mu$ and can be expanded in powers of ψ^μ

$$\Psi(x,\psi) = A + \psi^\mu A_\mu + \ldots + \frac{1}{D!}\psi^{\mu_1}\ldots\psi^{\mu_D}A_{\mu_1\ldots\mu_D} \tag{9}$$

where D is the space-time dimension and the coefficients $A_{\mu_1\ldots\mu_p}$ are antisymmetric tensors of rank p. The constraint $\hat{Q}\Psi = 0$ give us the following equations for the coefficients $A_{\mu_1\ldots\mu_p}$

$$\partial_{[\mu}A_{\mu_1\ldots\mu_p]} = 0\ ,\ p = 0,\ldots,D-1 \tag{10}$$

Due to the nilpotency of the constraint $\hat{Q}$ the condition $\hat{Q}\Psi = 0$ remains invariant if $\Psi \to \Psi + \hat{Q}\Lambda$ which implies the usual gauge transformation for antisymmetric tensors. If we take the space-time to be of the form $R \times \Sigma$ where Σ is a compact orientable $D-1$ dimensional manifold then the quantum states are square integrable functions on $H^p(\Sigma)$, the p^{th} cohomology class of Σ [7]. Due to the Poincaré duality $H^p(\Sigma)$ is isomorphic to $H^{D-1-p}(\Sigma)$ so that the coefficients of (9) are naturally split up into pairs of antisymmetric tensors of rank p and $D-1-p$. These are precisely the fields which are needed to build up the (abelian) topological field theories of the BF type and when D is odd and $p = \frac{D-1}{2}$ is also odd, of the Chern-Simons type [7].

We can also use the operator $\hat{Q}$ to write an action

$$S = \int dx \; d\psi \; \Psi \; \hat{Q} \; \Psi \tag{11}$$

from which we can recover the physical state condition. When integrated out in the fermionic coordinates we get a sum of all the actions of topological field theories in D dimensions.

It should be remarked that the last component of the wave function $A_{\mu_1 \ldots \mu_D}$ does not satisfy any equation and does not appear in the action (11). Therefore it is not a physical field and we can use the gauge transformation on Ψ to gauge away this component.

We will now show that it is possible to consider the action (1) in a gravitational background. We introduce the metric $g_{\mu\nu}$ through the vierbein e^a_μ so that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where $a, b, \ldots$ are tangent space indices. We also write the fermionic variables as $\psi^a = \psi^\mu e^a_\mu, \Pi_a = e^\mu_a \Pi_\mu, e^a_\mu e^\nu_a = \delta^\nu_\mu$. Then the action (1) in a gravitational background turns out to be

$$S = \int_{\tau_1}^{\tau_2} d\tau [P_\mu \dot{X}^\mu - i\Pi_a \dot{\psi}^a + i\lambda (P_\mu + i\Pi_a \psi^b \omega_{\mu b}{}^a) \psi^c e^\mu_c] \tag{12}$$

where $\omega_{\mu b}{}^a$ is the usual spin connection and P_μ was redefined as $P_\mu + i\Pi_a \psi^b \omega_{\mu b}{}^a$ so that the kinetic terms have the usual form.

It can be checked that the action (12) is invariant under the following worldline supersymmetry transformations

$$\begin{aligned}
\delta X^\mu &= -i\xi \psi^a e^\mu_a \\
\delta \psi^a &= -\frac{1}{4e}\xi(\dot{X}^\mu e^a_\mu + i\lambda\psi^a) + i\xi \Pi^b \psi^c e^\mu_b \omega_{\mu c}{}^a \\
\delta P_\mu &= -\frac{i}{4e}\xi(e^a_\mu \dot{\Pi}_a + \lambda P_\mu - \Pi_a \dot{X}^\nu e^c_{[\mu} \omega_{\nu]c}{}^a) + \\
& \quad i\xi\psi^a P_\nu \partial_\mu e^\nu_a - \xi \Pi_a \psi^b \psi^c \partial_\mu (e^\nu_c \omega_{\nu b}{}^a) \\
\delta \Pi_a &= -\xi e^\mu_a P_\mu + i\xi \Pi_b \psi^c \omega_{\mu[a}{}^b e^\mu_{c]} \\
\delta e &= \frac{i}{2}\xi\lambda \\
\delta\lambda &= \dot{\xi}
\end{aligned} \tag{13}$$

No term proportional to the curvature tensor is needed in the action.

So the classical theory is not topological since it couples to a gravitational background but when quantized it turns out to be a topological field theory. How this happens is not clear to us. If we had started with the action (12) we would get a constraint involving the gravitational field. However (first) quantization in curved spaces is still an open question since different authors arrive at different answers even for simple systems like a non-relativistic point particle moving on the surface of a sphere [8]. To treat a much more complex theory involving fermionic variables and constraints, such as given by (12), would require a better understanding of quantization on curved spaces. For lower dimensionality, however, the problem may be tractable but we leave this question to be discussed in a larger paper.

Finally we will give a superspace (superworldline) description of the topological spinning particle. The supercoordinates of the superspace are $Z^M = (\tau, \theta), M = (B, F)$ being a curved superspace index with bosonic and fermionic components B and F respectively. We introduce the supervierbein E_M^A where the tangent superspace index $A = (b, f)$ has bosonic and fermionic components b and f respectively. We also introduce a bosonic superfield $X^\mu(\tau, \theta)$ and a (worldline) fermionic superfield $\Pi_\mu(\tau, \theta)$ whose expansions in θ are

$$\begin{aligned} X^\mu(\tau,\theta) &= \tilde{X}^\mu + i\theta\tilde{\psi}^\mu \\ \Pi_\mu(\tau,\theta) &= \tilde{\Pi}_\mu + \theta\tilde{P}_\mu \end{aligned} \tag{14}$$

The action in superspace is then given by

$$S = -\frac{1}{2}\int d^2Z\ E^{-1}\ \Pi_\mu\ D_b\ X^\mu \tag{15}$$

where $E = detE_A^M$ and $D_b = E_b^M\partial_M$. It is invariant by general coordinate transformations in superspace as well as local tangent superspace rotations. To show the equivalence of the actions (15) and (1) we choose, in analogy with the spinning particle [9], the following torsion constraints

$$T_{ff}{}^b = -2i \tag{16}$$

with all the other components vanishing. This is the torsion of the flat superspace. This set of torsion constraints is invariant under super Weyl transformations

$$\begin{aligned} E_b^M &\rightarrow W^{-1}E_b^M + iW^{-\frac{3}{2}}(D_f W^{\frac{1}{2}})E_f^M \\ E_f^M &\rightarrow W^{-\frac{1}{2}}E_f^M \end{aligned} \tag{17}$$

with superparameter W. We can now choose a gauge in which the supervierbein differs from the flat superspace supervierbein

by a super Weyl transformation [9] with parameter

$$W(\tau,\theta) = \tilde{e} + i\theta\tilde{\lambda} \tag{18}$$

In this gauge the supervierbein has the following θ expansion

$$\begin{aligned} \tilde{E}_b^B &= \tilde{e}^{-1} - \frac{i}{2}\tilde{e}^{-2}\theta\tilde{\lambda} \\ \tilde{E}_b^F &= -\frac{1}{2}\tilde{e}^{-2}(\tilde{\lambda} + \theta\dot{\tilde{e}}) \\ \tilde{E}_f^B &= i\tilde{e}^{-\frac{1}{2}}\theta \\ \tilde{E}_f^F &= \tilde{e}^{-\frac{1}{2}} - \frac{i}{2}\tilde{e}^{-\frac{3}{2}}\theta\tilde{\lambda} \end{aligned} \tag{19}$$

This gauge choice is preserved under a general coordinate transformation in superspace, with parameter ξ^A, provided that

$$\xi^f = -\frac{i}{2}\mathcal{D}_f\xi^b \tag{20}$$

where $\mathcal{D}_f$ is the flat superspace covariant derivative $\mathcal{D}_f^2 = i\partial_\tau$. The θ expansion of ξ^A is

$$\begin{aligned} \xi^b &= \tilde{\epsilon} + i\theta\tilde{\rho} \\ \xi^f &= \tilde{\xi} + \theta\tilde{\eta} \end{aligned} \tag{21}$$

and the condition (20) imply in the following conditions $\tilde{\rho} = 2\tilde{\xi}$ and $\tilde{\eta} = -\frac{i}{2}\dot{\tilde{\epsilon}}$. A general coordinate transformation in this gauge reduces to

$$\begin{aligned} \delta\tilde{X}^\mu &= \tilde{\xi}\dot{\tilde{X}}^\mu - \frac{i}{2}(\mathcal{D}_f\tilde{\xi}^b)(\mathcal{D}_f\tilde{X}^\mu) \\ \delta\tilde{\Pi}_\mu &= \tilde{\xi}^b\dot{\tilde{\Pi}}_\mu - \frac{i}{2}(\mathcal{D}_f\tilde{\xi}^b)(\mathcal{D}_f\tilde{\Pi}_\mu) \\ \delta W &= (W\tilde{\xi}^b)^\cdot + \frac{i}{2}(\mathcal{D}_f W)(\mathcal{D}_f\tilde{\xi}^b) \end{aligned} \tag{22}$$

and by the following redefinition of the components in (14) and (18)

$$
\begin{aligned}
X^\mu = \tilde{X}^\mu \ , \ P_\mu &= -\frac{1}{2}e^{-\frac{1}{2}}\tilde{P}_\mu \\
\Pi_\mu = \tilde{\Pi}_\mu \ , \ \psi^\mu &= -\frac{1}{2}e^{-\frac{1}{2}}\tilde{\psi}^\mu \\
e = \tilde{e} \ , \ \lambda &= e^{\frac{1}{2}}\tilde{\lambda}
\end{aligned}
\qquad (23)
$$

and the components in (21) by

$$\epsilon = \tilde{\epsilon} \ , \ \xi = e^{\frac{1}{2}}\tilde{\xi} \qquad (24)$$

we get precisely the reparametrizations (2) and supersymmetry transformations (3).

References

[1] V.D.Gershern and V.I.Tkack, JETP Lett. 29 (1979) 288;
P.S.Howe, S.Penati, M.Pernici and P.K.Townsend, Phys.Lett 215 B (1988) 555

[2] G.Papadopoulos, Class.Quantum Grav. 6 (1989) 1745

[3] M.Pierri and V.O.Rivelles, Phys.Lett. 251 B (1990) 421; Phys.Rev. D 43 (1991) 2054

[4] P.S.Howe, S.Penati, M.Pernici and P.K.Townsend, Class.Quantum Grav. 6 (1989) 1125

[5] V.O.Rivelles and L.Sandoval Jr., Class.Quantum Grav. 8 (1991) 1605

[6] E.Witten, Nucl.Phys. B 311 (1988) 46; Comm.Math.Phys. 117 (1988) 353, 118 (1988) 411

[7] G.T.Horowitz, Comm.Math.Phys. 125 (1989) 417;
M.Blau and G.Thompson, Ann.Phys.(NY) 205 (1991) 130

[8] C.Grosche and F.Steiner, preprint DESY 90-115 (1990)
H.Kleinert, Phys.Lett. 236 B (1990) 315, Mod.Phys.Lett. 4 (1989) 2329;
R.Penrose, Proc.Roy.Soc. A 284 (1985) 159;
B.S.de Witt, Rev.Mod.Phys. 29 (1957) 377;
N.D.Birrel and P.C.W.Davies, **Quantum Fields in Curved Spaces** (Cambridge Univ.Press, 1982)

[9] L.Brink, S.Deser, B.Zumino, P.Di Vecchia and P.Howe, Phys.Lett. 64 B (1976) 435;

L.Brink, P.Di Vecchia and P.Howe, Nucl.Phys. B 118 (1977) 76