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A RELATION BETWEEN THE MASSIVE THIRRING MODEL AND THE GRASSMANN VALUED NONLINEAR SCHRÖDINGER EQUATION

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A Relation between the Massive Thirring Model and The Grassmann Valued Nonlinear Schrödinger Equation.

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We prove that the weakly coupled Thirring Model in the non Relativistic Limit is equivalent to an (anticommuting) variable obeying the Nonlinear Schrödinger Equation.

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The AKNS system[1] of second level has as a corresponding system of partial differential equations (p.d.e.) the pair of Nonlinear Schrödinger equations given by

$$\begin{cases} q_t = \frac{i}{2} q_{xx} - i\bar{q}qq \\ \\ \bar{q}_t = -\frac{i}{2}\bar{q}_{xx} + i\bar{q}q\bar{q} \end{cases} \tag{1}$$

This theory is integrable[2]; indeed if we consider the hamiltonian

$$H_1 = \int \sigma_2 dy = \int q(y)\bar{q}(y)dy, \tag{2}$$

the above system can be recast in the form

$$\frac{\partial}{\partial t}\cdot \begin{bmatrix} q(x) \\ \\ \bar{q}^T(x) \end{bmatrix} = i \begin{bmatrix} 0 & \left(-\frac{1}{2}\partial_{xx} + \sigma_2\right) I \\ \\ \left(-\frac{1}{2}\partial_{xx} + \sigma_2\right) I & 0 \end{bmatrix} \cdot \begin{bmatrix} +\bar{q}^T(x) \\ \\ -q(x) \end{bmatrix} \tag{3}$$

where we consider the functions q and $\bar{q}$ as two components spinors taking values in a Grassmann algebra and the structure above as being a symmetric Hamiltonian form of the nonlinear Schrödinger equations describing a fermionic system. As an example of a system with the structure (2), we consider the massive Thirring model given by the Lagrangian density

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi + \frac{1}{2}g\bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi \tag{4}$$

with

$$\Psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} \quad and \quad \bar{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0 = \begin{bmatrix} \psi_2^\dagger(x)\,\psi_1^\dagger(x) \end{bmatrix}. \tag{5}$$

The gamma matrices γ^μ are given by $\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,and $\gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

In terms of light-cone variables, the field equations obeyed by the Grassmann valued c-number functions $\psi_i(x)$ and $\psi_i^\dagger(x)$, $i = 1,2$ may be written as

$$\imath\sqrt{2}\partial_{\substack{- \\ +}}\psi_{\substack{1 \\ 2}}(x) = m\,\psi_{\substack{2 \\ 1}}(x) + 2\,g\,\psi_{\substack{1 \\ 2}}(x)\psi^\dagger_{\substack{2 \\ 1}}(x)\psi_{\substack{2 \\ 1}}(x) \tag{6}$$

$$\imath\sqrt{2}\partial_{\substack{- \\ +}}\psi^\dagger_{\substack{1 \\ 2}}(x) = -m\psi^\dagger_{\substack{2 \\ 1}}(x) - 2g\psi^\dagger_{\substack{2 \\ 1}}(x)\psi_{\substack{2 \\ 1}}(x)\psi^\dagger_{\substack{1 \\ 2}}(x) \tag{7}$$

where $x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}$, $\partial_- = \frac{\partial}{\partial x^-}$ and $\partial_+ = \frac{\partial}{\partial x^+}$. It is well known[3] that model has a infinite number of conservation laws, implying the integrability of the system. There exists an infinite number of conserved charges[4].

In the non relativistic limit, it is useful to consider the fields $\psi_i(x)$ written in terms of new fields $\phi_i(x)$ in the following form

$$\psi_i(x) = e^{\imath\frac{1}{2}mt}\varphi_i(x) \tag{8}$$

$$\psi_i^\dagger(x) = e^{-\imath\frac{1}{2}mt}\varphi_i^\dagger(x) \tag{9}$$

in terms of which the field equation, in a local coordinate system, are obtained from equations (6) to (7) as

$$\imath\partial_t\varphi_1(\sigma) = \frac{1}{2m}\partial_x^2\varphi_1(\sigma) + 2g\varphi_1^\dagger(\sigma)\varphi_2(\sigma)\varphi_1(\sigma) + \mathcal{O}(\frac{g}{m}) \tag{10}$$

$$\imath\partial_t\varphi_2(\sigma) = \frac{1}{2m}\partial_x^2\varphi_2(\sigma) + 2g\varphi_2^\dagger(\sigma)\varphi_1(\sigma)\varphi_2(\sigma) + \mathcal{O}(\frac{g}{m}) \tag{11}$$

Here we considered g (coupling fermions interaction constant) and m (mass of the system) as parameters satisfying the condition $\frac{g}{m} \to 0$. Introducting the quantity Υ such that the following properties are satisfied

$$\Upsilon(\sigma) = \bar{\Phi}(\sigma)\Phi(\sigma) = \varphi_1^\dagger(\sigma)\varphi_2(\sigma) + \varphi_2^\dagger(\sigma)\varphi_1(\sigma)$$

$$\Upsilon^\dagger(\sigma) = \Upsilon(\sigma)$$

the resulting field equations can be written in the form

$$\frac{\partial\Phi(\sigma)}{\partial t} = -\frac{\imath}{2m}\frac{\partial^2\Phi(\sigma)}{\partial x^2} - 2\imath g\Upsilon(\sigma)\Phi(\sigma) \tag{12}$$

$$\frac{\partial\bar{\Phi}^T(\sigma)}{\partial t} = \frac{\imath}{2m}\frac{\partial^2\bar{\Phi}^T(\sigma)}{\partial x^2} + 2\imath g\Upsilon(\sigma)\bar{\Phi}^T(\sigma) \tag{13}$$

to the two dimensional spinors $\Phi(\sigma)$ and its conjugate $\bar{\Phi}^T(\sigma)$ in the transpose form. We interpreted the equations above as the Nonlinear Schrödinger equation in the spinorial form.

Therefore, we conclude that the non-relativistic limit of the Thirring model corresponds to the nonlinear Schrödinger equation for (Grassmann-valued) nonrelativistic fermions, with a suitable redefinition of the space and time variables, that is, $x \to \frac{x}{\sqrt{mg}}$ and $t \to \frac{t}{g}$, in the limit $\frac{g}{m} \to 0$.

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