UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-943

1/N EXPANSION OF THE NON-LINEAR σ MODEL AND ITS RENORMALIZATION THROUGH STOCHATIC QUANTIZATION

J.C. Brunelli and M. Gomes
Instituto de Física, Universidade de São Paulo

Outubro/1991

$1/N$ Expansion of the Non–Linear σ Model and its Renormalization through Stochastic Quantization

J. C. Brunelli [(1)] and M. Gomes [(2)]

Instituto de Física da Universidade de São Paulo
Departamento de Física Matemática
CP 20516 – São Paulo – SP

Abstract

The $1/N$ expansion of non–linear σ model is considered in the framework of Parisi–Wu stochastic quantization. The expansion is implemented both in the Langevin approach and also in the functional–integral representation of stochastic processes. Whereas the first method quantization is possible only in a "tricky" way, the functional–integral approach is straightforward and thanks a BRST symmetry we show the renormalizability of the model in two and three dimensions.

This work was partially supported by (1) FAPESP and (2) CNPq.

1 Introduction

The stochastic quantization method (SQM) was devised by Parisi and Wu [1] (for a review see [2]) to avoid the introduction of gauge fixing terms in gauge theories. Although a lot of progress has been done in this direction, [3, 4], the SQM deserves an interest in itself as an alternative method for quantization. It is therefore important to reobtain well known results to understand the real power behind the method. From this point of view an interesting problem is the implementation of the $1/N$ expansion of field theories. We mention that the SQM was already used to study some properties of the large N limit, as, for example, the so called large N reduction [5]. In this article we study the $1/N$ expansion, [6], of the non–linear σ model, using the SQM. Classically, the model is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_a)^2 \tag{1}$$

in which the fields are subject to the constraint $\phi_a\phi_a = N/2f$. Perturbatively, one would use the constraint to eliminate one of the basic fields from the action to be used to generate Langevin dynamics, as was done in the references [7, 8].

In the usual (i. e., non stochastic) quantization approach, the simplest way to derive the $1/N$ expansion is to introduce a Lagrange multiplier field, σ, to enforce the constraint. The Lagrangian for the model then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 + \frac{\sigma}{\sqrt{N}}\left(\phi_a^2 - \frac{N}{2f}\right) \tag{2}$$

where a mass term for the ϕ_a field has been added. At the quantum level, the mass m will be adjusted so that σ has zero vacuum expectation value.

An apparent difficulty in Langevin's approach based on (2) is the lack of a restoring term, bilinear in the σ field. In section 2.,we overcome this problem by adding to the unperturbed Lagrangian and subtracting from the interacting Lagrangian a bilinear term with a kernel fixed by demanding that, at the static limit, in each order of $1/N$, only a finite number or diagrams contribute. With this proviso, higher order corrections can be calculated up to ultraviolet divergencies which must be removed by a renormalization prescription. The discussion of this problem, however, is easier in the functional approach, [9, 10] where techniques similar to those used in gauge theories can be employed, [11]. Because the phase space has dimension $n+2$, n being the space-time dimension, the stochastic diagrams have degree of superficial divergence higher than usual. Fortunately, as discussed in section 3., a BRST symmetry puts strong restrictions to the form of the allowed counterterms. However these restrictions are not enough to establish renormalizability. To achieve that, we exploit some additional Ward identities related to the basic constraint, $\phi_a\phi_a = \frac{N}{2f}$, which the fields should obey. For $2 \leq n < 4$ we are then able to prove the renormalizability of the model.

In the conclusions, section 4., the model is rewritten in an explicitly supersymmetric form. This is a very important step, guarantying the existence of the static limit, through the mechanism of dimensional reduction of Parisi Sourlas, [12, 13, 14], adapted to the SQM, [15, 16].

2 The $1/N$ Expansion of Non–Linear σ Model in the Langevin Approach

In quantum field theory the basic objects are the Green functions given, in Euclidian space, by

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle \equiv \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, \mathrm{e}^{-S[\phi]}}{\int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]}} \tag{3}$$

The idea of Parisi and Wu [1], to obtain these Green functions, was to incorporate an additional coordinate τ (called fifth or fictitious time) in the fields, $\phi(x) \to \phi(x,\tau)$, and to postulate a dynamics for the field, given by a stochastic differential equation (Langevin equation)

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{d}{2}\frac{\delta S[\phi]}{\delta\phi(x,\tau)} + \eta(x,\tau) \tag{4}$$

where S is the Euclidian action given by

$$S = \int d^n x\, d\tau'\, \mathcal{L}(\phi(x,\tau'), \partial_\mu\phi(x,\tau')), \tag{5}$$

d is the diffusion coefficient and $\eta(x,\tau)$ is a Gaussian white noise with correlations

$$\langle\eta(x,\tau)\rangle_\eta = 0 \tag{6}$$

$$\langle\eta(x_1,\tau_1)\,\eta(x_2,\tau_2)\rangle_\eta = d\,\delta^n(x_1-x_2)\,\delta(\tau_1-\tau_2) \tag{7}$$

$$\langle\eta(x_1,\tau_1)\cdots\eta(x_{2k+1},\tau_{2k+1})\rangle_\eta = 0 \tag{8}$$

$$\langle\eta(x_1,\tau_1)\cdots\eta(x_{2k},\tau_{2k})\rangle_\eta = \sum_{\substack{pair\\ combinations}} \prod_{pairs} \langle\eta(x_i,\tau_i)\,\eta(x_j,\tau_j)\rangle_\eta \tag{9}$$

The correlations of functions of the ϕ field are given by averages over the noise η with a Gaussian measure

$$\langle\cdots\rangle_\eta = \frac{\int \mathcal{D}\eta \cdots \exp\Big\{-1/2d \int d^n x\, d\tau\, \eta^2(x,\tau)\Big\}}{\int \mathcal{D}\eta \exp\Big\{-1/2d \int d^n x\, d\tau\, \eta^2(x,\tau)\Big\}} \tag{10}$$

where the dots stands for any product of functions of ϕ. The basic recipe of stochastic quantization is to solve (4) subject to a initial condition $\phi_0 = \phi(x, \tau = \tau_0)$ and to construct the correlations functions of ϕ_η,

$$\langle \phi_\eta(x_1,\tau_1)\phi_\eta(x_2,\tau_2)\cdots\phi_\eta(x_n,\tau_n)\rangle_\eta \tag{11}$$

It is then possible to demonstrate [1,17] that at large and equal fifth times (i. e. $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$) these correlations tend to the corresponding Green functions of the quantum field theory. Precisely,

$$\lim_{\tau\to\infty}\langle \phi_\eta(x_1,\tau)\phi_\eta(x_2,\tau)\cdots\phi_\eta(x_n,\tau)\rangle_\eta = \langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{12}$$

As a test to this approach we want to analyse the 1/N expansion for the non linear sigma model described by (2). The first idea is to consider Langevin equation for both ϕ and σ. However, the Langevin equation for σ does not possess a restoring term leading to divergencies in the equilibrium. This same behaviour is to be noted in gauge theories where the Langevin equation for the longitudinal part of the gauge potential does not have a damping term. In that case the problem can be circumvented by the use of the stochastic gauge fixing [3]. In our case we can handle the problem considering instead of (2) a modified Lagrangian proposed by Lowenstein and Speer [18]. It is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 + \frac{1}{2}\sigma\,\Sigma(-\Box)\,\sigma$$

$$\mathcal{L}_{int} = \frac{1}{\sqrt{N}}\sigma\phi_a^2 - \frac{1}{2}\sigma\,\Sigma(-\Box)\,\sigma \tag{13}$$

where Σ fixed so that at the static limit only a finite number of diagrams contribute in each order of $1/N$. Explicitly,

$$\Sigma(p^2) = -2\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m^2)\left[(p-k)^2+m^2\right]}. \tag{14}$$

In momentum space the Langevin equations associated to (13) are

$$\Big[\frac{\partial}{\partial t} + \frac{d_\phi}{2}(k^2+m^2)\Big]\phi_a = \eta_a - \frac{d_\phi}{\sqrt{N}}\int\frac{d^n p}{(2\pi)^n}\phi_a(p,t)\sigma(k-p,t) \tag{15}$$

and

$$\begin{aligned}\Big[\frac{\partial}{\partial t} + \frac{d_\sigma}{2}\Sigma(k^2)\Big]\sigma(k,t) &= \eta_\sigma + \frac{d_\sigma}{2}\sigma(k,t)\Sigma(k^2)\\ &\quad - \frac{d_\sigma}{2\sqrt{N}}\int\frac{d^n p}{(2\pi)^n}\phi_a(p,t)\phi_a(k-p,t)\end{aligned} \tag{16}$$

where d_ϕ and d_σ are diffusion coefficients. Their precise values are not important since the equilibrium limit is independent of them.

We will solve these equations taking as initial conditions that the fields ϕ_a and σ vanish for t tending to minus infinity. The free field Green functions, obtained by replacing the right hand side of (16) by delta functions, are

$$G_{ab}(k,t) = \delta_{ab}e^{-\frac{d_\phi}{2}(k^2+m^2)t}\theta(t) \tag{17}$$

$$G_\sigma(k,t) = e^{-\frac{d_\sigma}{2}\Sigma(k^2)t}\theta(t) \tag{18}$$

so that the free field propagators are

$$D_{ab}(k;t,t') = \langle\phi_a(k,t)\phi_b(k',t')\rangle = (2\phi)^n\delta_{ab}\delta^n(k+k')\frac{e^{-\frac{d_\phi}{2}(k^2+m^2)|t-t'|}}{k^2+m^2} \tag{19}$$

$$D_\sigma(k;t,t') = \langle\sigma(k,t)\sigma(k',t')\rangle(2\phi)^n\delta^n(k+k')\frac{e^{-\frac{d_\sigma}{2}\Sigma(k^2)|t-t'|}}{\Sigma(k^2)} \tag{20}$$

Solving Langevin equations by iteration, it turns out to be convenient to introduce a graphical notation as indicated in fig. 1. One can then verify that graphs containing the insertion of the bilinear vertice are cancelled by graphs containing one loop diagrams of fig. 2. as subgraphs (this is necessary to have a finite number of graphs in each order of $\frac{1}{N}$). Indeed, we have the following contributions to the graphs listed in fig. 2.

$$G_1 + G_2 = \frac{d_\sigma}{2}\Sigma(k^2)\int_{-\infty}^{t'} d\tau_1\, e^{-\frac{d_\sigma}{2}\Sigma(k^2)(t'-\tau_1)}\frac{e^{-\frac{d_\sigma}{2}\Sigma(k^2)|t-\tau_1|}}{\Sigma(k^2)} \quad + \quad (t \to t') \tag{21}$$

$$\begin{aligned} G_3 + G_5 = {} & \frac{d_\phi d_\sigma}{2}\int\frac{d^n k_1}{(2\pi)^n}\int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2\, e^{-\frac{d_\sigma}{2}\Sigma(k^2)(t-\tau_1)}\frac{e^{-\frac{d_\sigma}{2}\Sigma(k^2)|t'-\tau_2|}}{\Sigma(k^2)} \\ & \times \frac{e^{-\frac{d_\phi}{2}[(k-k_1)^2+m^2]|\tau_1-\tau_2|}}{(k-k_1)^2+m^2} e^{-\frac{d_\phi}{2}(k_1^2+m^2)(\tau_1-\tau_2)} \quad + \quad (t \rightleftharpoons t', \tau_2 \rightleftharpoons \tau_1) \end{aligned} \tag{22}$$

$$G_4 + G_6 = G_3 + G_5 \quad \text{with} \quad (k_1 \rightleftharpoons k - k_1) \tag{23}$$

$$\begin{aligned} G_7 \;=\; & d_\sigma^2\int\frac{d^n k_1}{(2\pi)^n}\int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2\, e^{-\frac{d_\sigma}{2}\Sigma(k^2)(t-\tau_1)} e^{-\frac{d_\sigma}{2}\Sigma(k^2)(t'-\tau_2)} \\ & \times \frac{e^{-\frac{d_\phi}{2}(k_1^2+m^2)|\tau_1-\tau_2|}}{k_1^2+m^2}\frac{e^{-\frac{d_\phi}{2}[(k-k_1)^2+m^2]|\tau_1-\tau_2|}}{(k-k_1)^2+m^2} \end{aligned} \tag{24}$$

The static limit is reached by setting $t = t'$. We get

$$G_1 + G_2 = \frac{1}{\Sigma(k^2)} \tag{25}$$

$$\begin{aligned} G_3 + \cdots \;+\; G_7 = {} & 2\int\frac{d^n k_1}{(2\pi)^n}\,\frac{1}{\Sigma(k^2)\left[d_\sigma/d_\phi\,\Sigma(k^2) + (k-k_1)^2 + k_1^2 + 2m^2\right]} \\ & \times \left\{\frac{d_\sigma/d_\phi}{(k_1^2+m^2)\left[(k-k_1)^2+m^2\right]} + \frac{1}{\Sigma(k^2)\left[(k-k_1)^2+m^2\right]} \right. \\ & \left. + \frac{1}{\Sigma(k^2)\,(k_1^2+m^2)}\right\} = -\frac{1}{\Sigma(k^2)} \end{aligned} \tag{26}$$

so that the sum of all these contributions vanish, as promised. Therefore both graphs with the bilinear insertion and graphs with the one loop diagrams of fig. 2 can be omitted once and for all.

We are in a position to discuss higher order corrections and the inevitable ultraviolet divergencies. However, as the stochastic diagrams are very different from the ordinary Feynman graphs, the discussion can become somehow cumbersome. It is advantageous to use the functional integral approach that keeps a closer resemblance with the usual methods. Nonetheless, as we will see shortly, there will be additional divergencies, not present in the usual formalism.

3 The $1/N$ Expansion of the Non–Linear σ Model in the Functional–Integral Approach

Following Gozzi [9] and Nakano [10] we will derive a functional generator $Z[J]$ for the correlations functions (11), i.e.,

$$\langle \phi_\eta(x_1,\tau_1)\cdots\phi_\eta(x_n,\tau_n)\rangle_\eta = \left.\frac{\delta^n Z[J]}{\delta J(x_1,\tau_1)\cdots\delta J(x_n,\tau_n)}\right|_{J=0} \tag{27}$$

expressed in a functional integral form over the fields. To construct the functional generator for the non linear sigma model we shall consider Langevin's equation and the constraint. We have

$$\begin{aligned} \dot\phi_a \;=\; & -\frac{d}{2}\left(\frac{\delta S[\phi]}{\delta\phi_a} + \frac{2}{\sqrt{N}}\sigma\phi_a\right) + \eta_a \\ & \phi_a^2 - \frac{N}{2f} = 0 \end{aligned} \tag{28}$$

with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 \tag{29}$$

As shown in [11], there is a BRST symmetry in the functional representation for stochastic processes. This symmetry will allow us to analyse the renormalization of the model in a systematic way. Let us introduce the simplified notation

$$\xi = \begin{pmatrix} \phi \\ \frac{\alpha}{\sqrt{N}} \end{pmatrix}, \quad \nu = \begin{pmatrix} \eta \\ 0 \end{pmatrix}, \quad K = \begin{pmatrix} J \\ 0 \end{pmatrix} \tag{30}$$

where the first row of each matrix is actually a N component column matrix. Equations (28) can be rewritten now as

$$F(\xi) = \nu \tag{31}$$

where

$$\begin{cases} F_a(\phi, \sigma) = \dot{\phi}_a + \frac{d}{2}\left(\frac{\delta S}{\delta \phi_a} + \frac{2}{\sqrt{N}}\sigma\phi_a\right) & \text{for } a = 1, \ldots, N \\ F_a(\phi) = \phi_a^2 - \frac{N}{2f} \equiv F & \text{for } a = N+1 \end{cases} \tag{32}$$

The generating functional is then

$$Z[K] = \left\langle e^{\int_{-\infty}^{+\infty} d^n x d\tau \, K_a \xi_a^\nu} \right\rangle_\nu \tag{33}$$

where ξ_a^ν is the solution of $F(\xi) = \nu$. For our Gaussian process

$$Z[K] = \mathcal{N} \int \mathcal{D}\nu \, e^{\int_{-\infty}^{+\infty} d^n x d\tau \, K_a \xi_a^\nu - \frac{1}{2d}\int_{-\infty}^{+\infty} d^n x d\tau \, \nu_a^2} \tag{34}$$

where $\mathcal{D}\nu = \prod_a \mathcal{D}\eta_a$. This equation can be rewritten as

$$Z[K] = \mathcal{N} \int \mathcal{D}\nu \mathcal{D}\xi \, \delta(\xi_a - \xi_a^\nu) e^{\int_{-\infty}^{+\infty} d^n x d\tau \, K_a \xi_a - \frac{1}{2d}\int_{-\infty}^{+\infty} d^n x d\tau \, \nu_a^2} \tag{35}$$

so that, because of

$$\delta(\xi_a - \xi_a^\nu) = \delta(F_a(\xi) - \nu_a) \left\| \frac{\delta F_a}{\delta \xi_b} \right\| \tag{36}$$

we have

$$Z[K] = \mathcal{N} \int \mathcal{D}\nu \mathcal{D}\xi \, \delta(F_a(\xi) - \nu_a) \left\| \frac{\delta F_a}{\delta \xi_b} \right\| e^{-\int d^n x d\tau \left(\frac{1}{2d}\nu_a^2 - K_a \xi_a\right)} \tag{37}$$

Using now the identities

$$\delta(F_a(\xi) - \nu_a) = \int \mathcal{D}\beta \, e^{-\int d^n x d\tau \, \beta_a (F_a(\xi) - \nu_a)} \tag{38}$$

$$\left\| \frac{\delta F_a}{\delta \xi_b} \right\| = \int \mathcal{D}D\mathcal{D}\overline{D} \, e^{\int d^n x d\tau d^n x' d\tau' \, \overline{D}_a(x,\tau) \frac{\delta F_a(x,\tau)}{\delta \xi_b(x',\tau')} D_b(x',\tau')} \tag{39}$$

where (D and $\overline{D}$ are Grassmann fields)

$$\beta = \begin{pmatrix} \lambda \\ \frac{\alpha}{\sqrt{N}} \end{pmatrix}, \quad \overline{D} = \begin{pmatrix} \overline{C} \\ \frac{\bar{c}}{\sqrt{N}} \end{pmatrix} \quad \text{e} \quad D = \begin{pmatrix} C \\ \frac{c}{\sqrt{N}} \end{pmatrix} \tag{40}$$

we have (for notational simplicity we will write x instead of (x, τ))

$$\begin{aligned} Z[K] = \mathcal{N} \int \mathcal{D}\nu \mathcal{D}\xi \mathcal{D}\beta \mathcal{D}D\mathcal{D}\overline{D} \, \exp\Big\{ - \int dx \Big[\beta_a (F_a(\xi) - \nu_a) \\ - \int dy \, \overline{D}_a(x) \frac{\delta F_a(x)}{\delta \xi_b(y)} D_b(y) - K_a \xi_a + \frac{1}{2d}\nu_a^2 \Big] \Big\} \end{aligned} \tag{41}$$

Integrating in $\nu_a = \begin{pmatrix} \eta_a \\ 0 \end{pmatrix}$, we finally obtain

$$\begin{aligned} Z[K] = \mathcal{N} \int \mathcal{D}\xi \mathcal{D}\beta \mathcal{D}D\mathcal{D}\overline{D} \, \exp\Big\{ - \int dx \Big[\beta_a F_a(\xi) - \frac{d}{2}\tilde{\beta}_a^2 \\ - \int dy \, \overline{D}_a(x) \frac{\delta F_a(x)}{\delta \xi_b(y)} D_b(y) \Big] + \int dx \, K_a \xi_a \Big\} \end{aligned} \tag{42}$$

with

$$\tilde{\beta} = \mathcal{O}\beta \quad \text{where} \quad \mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{43}$$

It is important to observe, [11], that the effective action in (42),

$$\mathcal{S}(\xi, \beta, D, \overline{D}) = -\frac{d}{2}\int dx \, \tilde{\beta}_a^2 + \int dx \, \beta_a F_a(\xi) - \int dx dy \, \overline{D}_a(x) M_{ab}(x,y) D_b(y) \tag{44}$$

where

$$M_{ab}(x,y) = \frac{\delta F_a(x)}{\delta \xi_b(y)} \tag{45}$$

is invariant under the following transformation ($\bar{\epsilon}$ is a Grassmann variable)

$$\begin{aligned}\delta\xi_a(x) &= \bar{\epsilon}D_a(x)\\ \delta D_a(x) &= 0\\ \delta\overline{D}_a(x) &= \bar{\epsilon}\beta_a(x)\\ \delta\beta_a(x) &= 0\end{aligned} \qquad (46)$$

which is nilpotent, $\delta^2 = 0$, and is called BRST by analogy with the corresponding symmetry found in gauge theories. To derive the Ward identities associated to this symmetry we add external sources $\overline{\vartheta}$, ϑ and M for D, $\overline{D}$ and β respectively. We have then

$$\begin{aligned}Z[K,M,\vartheta,\overline{\vartheta}] &= \mathcal{N}\int \mathcal{D}\xi\mathcal{D}\beta\mathcal{D}D\mathcal{D}\overline{D}\,\exp\Big\{-S(\xi,\beta,D,\overline{D})+\\ &+\int dx\,\left(K_a\xi_a + M_a\beta_a + \overline{\vartheta}_a D_a + \overline{D}_a\vartheta_a\right)\Big\}\end{aligned} \qquad (47)$$

We now change the integration variables in the way dictated by (46). If the measure is invariant by the BRST symmetry, we obtain

$$\int dx\,\left(K_a\frac{\delta Z}{\delta\overline{\vartheta}_a} + \frac{\delta Z}{\delta M_a}\vartheta_a\right) = 0 \qquad (48)$$

Defining the generating functional for the 1PI functions by

$$\Gamma[\xi,\beta,D,\overline{D}] = W[K,M,\vartheta,\overline{\vartheta}] - \int dx\,\left(K_a\xi_a + M_a\beta_a + \overline{\vartheta}_a D_a + \overline{D}_a\vartheta_a\right) \quad (49)$$

where W is the generating functional for the connected functions, we find that

$$\int dx\,\left(\frac{\delta\Gamma}{\delta\xi_a}D_a + \beta_a\frac{\delta\Gamma}{\delta\overline{D}_a}\right) = 0 \qquad (50)$$

Assuming that a regularization scheme can be devised so that the BRST symmetry is preserved under renormalization, we obtain that the renormalized effective action still satisfy the above equation. Suppose moreover that

this renormalized effective action has a form similar to (44),

$$S_R = -\int dxdy\, M^R_{ab}(\xi)\overline{D}_a D_b + \Sigma(\xi,\beta) \qquad (51)$$

where $M^R_{ab} = 0$ for $a = b = N+1$. From equation (50) it follows that, [11],

$$\frac{\delta M^R_{ab}(x,y)}{\delta\xi_c(z)} - \frac{\delta M^R_{ac}(x,z)}{\delta\xi_b(y)} = 0 \qquad (52)$$

$$\frac{\delta\Sigma}{\delta\xi_a(x)} = \int dy\,\beta_b(y)M^R_{ba}(y,x) \qquad (53)$$

and therefore

$$M^R_{ab} = \frac{\delta F^R_a(x)}{\delta\xi_b(y)} \qquad (54)$$

where F^R_a depends only on ξ.

The solution of (53) can be written as

$$\Sigma = \int dx\,\beta_a(x)F^R_a(x) - W^R(\beta) \qquad (55)$$

so that, replacing it into (51), we get

$$S_R = -\int dxdy\,\frac{\delta F^R_a(x)}{\delta\xi_b(y)}\overline{D}_a(x)D_b(y) + \int dx\,\beta_a F^R_a(\xi) - W^R(\beta) \qquad (56)$$

Notice that F^R_{N+1} depends only on ϕ and not on σ, because M^R_{ab} vanishes for $a = b = N+1$. We see that this BRST symmetry puts strong restrictions to the form of possible counterterms. Thus, for example, monomials containing powers of the σ field only, are not allowed if the BRST symmetry is preserved under renormalization. However, the BRST symmetry by itself does not guarantee the renormalizability of the model; it still remains to prove that $F^R(\xi)$ depends on ξ in the way dictated by (32) and that $W^R(\beta)$ is quadratic in $\tilde{\beta}$.

Using power counting, let us establish that the renormalized effective action has indeed the form (51). At this point it is convenient to go back to the original notation, writing the effective action in terms of the components of the fields ξ, β, $\overline{D}$ and D. We have

$$\mathcal{S}(\phi,\sigma,\lambda,\alpha,\overline{C},C,\overline{c},c)=\int d^nxd\tau\left[-\frac{d}{2}\lambda_a^2+\lambda_aF_a(\phi,\sigma)+\frac{\alpha}{\sqrt{N}}F(\phi)\right]$$
$$-\int d^nxd\tau d^nx'd\tau'\left[\overline{C}_a\frac{\delta F_a}{\delta\phi_b}C_b+\overline{C}_a\frac{\delta F_a}{\delta\sigma}c+\frac{1}{\sqrt{N}}\overline{c}\frac{\delta F}{\delta\phi_b}C_b\right] \tag{57}$$

and, using (32), we obtain

$$\begin{aligned}\mathcal{S} \;=\; &\int d^nxd\tau\left\{-\frac{d}{2}\lambda_a^2+\lambda_a\left[\dot{\phi}_a+\frac{d}{2}(-\Box+m^2)\phi_a+\frac{d}{\sqrt{N}}\sigma\phi_a\right]\right.\\&+\frac{\alpha}{\sqrt{N}}\left(\phi_a^2-\frac{N}{2f}\right)-\overline{C}_a\left[\frac{d}{d\tau}+\frac{d}{2}(-\Box+m^2)\right.\\&\left.\left.+\frac{d}{\sqrt{N}}\sigma\right]C_a-\frac{d}{\sqrt{N}}\overline{C}_a\phi_a c-\frac{2}{\sqrt{N}}\overline{c}\phi_aC_a\right\}\end{aligned} \tag{58}$$

The Feynman rules adequate to the $1/N$ expansion based on this Lagrangian may be obtained by summing bubble diagrams and inverting the resulting quadratic form. They are depicted on fig. 3. Using that figure we verify that $\Gamma_{\sigma\sigma}=0$ and

$$\Gamma_{\alpha\alpha}(\beta p,\beta^2\omega)\;\rightarrow\;\beta^{n-6} \tag{59}$$
$$\Gamma_{\alpha\sigma}(\beta p,\beta^2\omega)\;\rightarrow\;\beta^{n-4} \tag{60}$$
$$\Gamma c\overline{c}(\beta p,\beta^2\omega))\;\rightarrow\;\beta^{n-4} \tag{61}$$

as β tends to infinity. From those results we may compute the degree of superficial divergence of a generic graph γ. We have

$$\delta=DL-\sum_i d_in_i+\sum_v D_v \tag{62}$$

where n_i = number of internal lines of the type i (i. e., lines associated with a given propagator), d_i is the ultraviolet degree of the propagator of the type i ($G_i(\beta p,\beta^2\omega)\rightarrow\beta^{-d_i}$ as $\beta\rightarrow\infty$); L is the number of loops in the graph, D_v = number of derivatives at the vertex v and $D=n+2$ where n is the dimension of the space-time. This formula can be put into a more manageable form by using the usual relations between number of loops, number of internal lines, external lines and of vertices belonging to the graph. In this way we obtain

$$\delta=D-\sum_i[A_i]N_{A_i}-\sum_v(D-[\mathcal{L}_v]) \tag{63}$$

where

$$[A_i]=\frac{D-d_{A_i}}{2}\quad\text{and}\quad[\mathcal{L}_v]=D_v+\sum_i[A_i]\nu_v^{A_i} \tag{64}$$

with d_{A_i} = asymptotic behaviour of the A_i propagator for large momenta and ω's ($\beta^{-d_{A_i}}$ whenever $k\rightarrow\beta k$, $\omega\rightarrow\beta^2\omega$ and $\beta\rightarrow\infty$); N_{A_i} = number of external lines associated to the A_i field (in the case of a line associated to the contraction of two different fields, i.e., mixed propagators, one should count ends of external lines directly attached to the graph); D_v = number of derivatives at the vertex v; $\nu_v^{A_i}$ = number of lines associated to A_i joining at the vertex v; $[A_i]$ = canonical dimension of the A_i field; $[\mathcal{L}_v]$ = operator dimension of the vertex v and $\sum_{v(i)}$ = sum over all vertices of G.

As a general observation concerning the present approach, notice that due to the effective increase by two in the dimension of the phase space many more superficially divergent diagrams are generated. We shall see shortly how additional Ward identities can be used to overcome this problem.

The dimensions of the basic fields are easily obtained using (64) and (58),

$$\begin{array}{ll} [\phi] = \frac{n-2}{2} & [\sigma] = 2 \\ [\lambda] = \frac{n+2}{2} & [\alpha] = 4 \\ [\overline{C}] + [C] = n & [\overline{c}] + [c] = 6 \end{array} \qquad (65)$$

It follows then that for all vertices $[\mathcal{L}_v] \leq D$ so that in the less favorable situation (63) gives

$$\delta = D - \sum_i [A_i] N_{A_i} \qquad (66)$$

Let us now try to satisfy the requirements of the BRST symmetry. Firstly, we should guarantee that the renormalized action, S_R, does not depend on $\overline{c}c$. From (66) this implies that

$$[\overline{c}] + [c] = 6 > n + 2, \qquad (67)$$

i. e. $n < 4$. Similarly, in order for S_R to be at most quadratic in $\overline{C}C$, $\overline{C}c$ and $\overline{c}C$ we would need

$$\left.\begin{array}{r} 2([\overline{C}] + [C]) = 2n \\ 2([\overline{C}] + [c]) = n + 6 \end{array}\right\} > n + 2 \qquad (68)$$

which is obeyed if $n > 2$. On the other hand, the coefficient of $\overline{C}C$, $\overline{C}c$ and $\overline{c}C$ will depend only on σ and ϕ if

$$\left.\begin{array}{r} [\overline{C}] + [C] + [\lambda] = \frac{3n+2}{2} \\ [\overline{C}] + [C] + [\alpha] = n + 4 \\ [\overline{C}] + [c] + [\lambda] = n + 4 \\ [\overline{C}] + [c] + [\alpha] = \frac{n+14}{2} \end{array}\right\} > n + 2 \qquad (69)$$

which leads to $n > 2$. We are therefore limited to the interval $2 < n < 4$. In that interval we can go from (51) to (56) without having to verify explicitly the vanishing of some of the counterterms. Thus, for example, graphs with two external σ lines have degree of superficial divergence $\delta = n - 2$ leading

apparently to a non renormalizable theory. However, a σ^2 counterterm is allowed by (51) but not by (56), implicating that it must vanish. Although we have proven that, as consequence of the BRST symmetry, the renormalized action has the form (56) we still have to prove that $W^R(\beta)$ is quadratic in λ and that $\lambda_a F_a^R(\phi, \sigma)$ and $\alpha F^R(\phi)$ are renormalized versions of (32). The first condition, namely that W^R is quadratic in λ, is satisfied if

$$\left.\begin{array}{r} 2[\alpha] = 8 \\ [\alpha] + [\lambda] = \frac{n+10}{2} \end{array}\right\} > n + 2 \qquad (70)$$

i. e., $n < 6$ which is compatible with the interval determined by the BRS symmetry. Concerning $\lambda_a F_a^R$ and F^R notice that at $n = 3$ it would be possible to generate counterterms of the type

$$(\lambda_a \phi_a) \phi_b^2 \quad \text{e} \quad (\lambda_a \phi_a) \phi_b^2 \phi_c^2 \qquad (71)$$

with $\delta = 1$ and $\delta = 0$ respectively. However, as indicated in fig. 4 and fig. 5, it is easy to verify that such counterterms do cancel. Notice that the contributions of graphs of type (5c) are not relevant if one considers only correlations of physical fields. Indeed, the corresponding graphs do not have neither C or $\bar{C}$ as external fields. Therefore, the lines associated to these fields must form closed loops. Noting now that the propagators depend on the fifth time in the way

$$\langle C(k,\tau) \overline{C}(k',\tau') \rangle \;\propto\; \theta(\tau - \tau') \qquad (72)$$

$$\langle c(k,\tau) \overline{c}(k',\tau') \rangle \;\propto\; \delta'(\tau - \tau') \qquad (73)$$

we arrive to the conclusion that these graphs will vanish when integrated on the fifth time. Thus ghosts field do not contribute to the correlation function of other fields. We arrive to the conclusion that for $n = 3$ the non linear sigma model is renormalizable in the $1/N$ expansion. The same is true in $2+\epsilon$ dimension but, letting $\epsilon \to 0$, we find new types of divergencies. Firstly, there are divergencies associated to graphs with four external ghosts(i. e. C and $\overline{C}$ lines). In this case, we do not add the corresponding counterterms because this would destroy the BRST symmetry of the renormalized Lagrangian. Since, as mentioned earlier, closed loops of ghosts fields vanish, the physical sector consisting of the correlations functions of the ϕ_a fields will be free from this type of divergence. Besides that, new divergencies arise because the field ϕ_a has zero dimension at $n=2$ and so the degree of superficial divergence does not change if the number of external ϕ_a lines is increased. Notice however that the divergent graphs must have at least one external line different from those associated to the ϕ_a field, in accordance with (56). These diagrams would generate counterterms containing powers of ϕ_a^2. If the divergence is only logarithimic, we could use the same mechanism indicated in fig 4b and 4c, to show that there is a cancellation of the corresponding contributions. Nonetheless, this does not exhaust all the possibilities. In the case of a graph with one external λ_a line and three external ϕ_a lines the graph is quadratically divergent and in principle would need couterterms like $\lambda_a\phi_a\phi^2$ and $\lambda_a\phi_a\partial_\mu\phi_b\partial^\mu\phi_b$. To rule out the second possibility we should use the identity in fig. 4a. We concluded that also in two dimensions the model will remain renormalizable as before

We now integrate in $\overline{C}$, C, $\bar{c}$ and c (or $\overline{D}$ and D in (47)) resulting

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\mathcal{D}\sigma\mathcal{D}\lambda\mathcal{D}\alpha \,\det M \exp\Big\{-\int d^n x dt\,\Big[-\frac{d}{2}\lambda_a^2 + \lambda_a\Big(\dot\phi_a + \frac{d}{2}(-\Box + m^2)\phi_a + \frac{d}{\sqrt{N}}\sigma\phi_a\Big) + \frac{\alpha}{\sqrt{N}}\left(\phi_a^2 - \frac{N}{2f}\right) - J\phi\Big]\Big\} \tag{74}$$

where (as can be seen from (45))

$$\det M = \begin{vmatrix} \frac{\delta F_a}{\delta\phi_b} & \sqrt{N}\frac{\delta F_a}{\delta\sigma} \\ \frac{\delta F}{\delta\phi_b} & \sqrt{N}\frac{\delta F}{\delta\sigma} \end{vmatrix} \tag{75}$$

It is possible to show that, [7, 9, 11],

$$\det M \sim \exp\left\{\frac{d}{4}\delta^n(0)\int d^n x d\tau\,\sigma(x,\tau)\right\} \tag{76}$$

which plays the role of a counterterm to cancel some divergencies of the perturbative series, [20]. Alternatively, using dimensional regularization one can disregard (76) because $\delta^n(0) \equiv (\int d^n k)_R = 0$ [7, 11, 20], where R indicates that the integral is dimensionally regularized. From (74), integrating in λ, we obtain

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\mathcal{D}\sigma\mathcal{D}\alpha \,\det M \exp\Big\{-\int d^n x dt\,\Big[\frac{1}{2d} \times \left(\dot\phi_a + \frac{d}{2}(-\Box + m^2)\phi_a + \frac{d}{\sqrt{N}}\sigma\phi_a\right)^2 + \frac{1}{\sqrt{N}}\alpha\left(\phi_a^2 - \frac{N}{2f}\right) - J\phi\Big]\Big\} \tag{77}$$

and, discarding the contribution of the determinant, we arrive at

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi\mathcal{D}\sigma\mathcal{D}\alpha \exp\Big\{-\int d^n x dt\,\Big[\frac{1}{2d} \times\phi_a\Big(K^\dagger K + \frac{d}{\sqrt{N}}\sigma K + \frac{d}{\sqrt{N}}K^\dagger\sigma + \frac{d^2}{N}\sigma^2 + \frac{2d}{\sqrt{N}}\alpha\Big)\phi_a - \frac{\sqrt{N}}{2f}\alpha - J_a\phi_a\Big]\Big\} \tag{78}$$

where

$$K = \frac{d}{dt} + \frac{d}{2}(-\Box + m^2) \tag{79}$$

$$K^\dagger = -\frac{d}{dt} + \frac{d}{2}(-\Box + m^2) \tag{80}$$

This expression could be used as a starting point to do $1/N$ calculations. The advantage over (47) is that it is more economic, employing less auxiliary fields.

Integrating over the ϕ_a fields one gets

$$Z[J] = \mathcal{N} \int \mathcal{D}\sigma \mathcal{D}\alpha \exp\Big\{-\mathcal{A}_{ef} - \frac{d}{2}\int d^n x dt\, J_a D^{-1} J_a\Big\} \tag{81}$$

where

$$D = K^\dagger K + \frac{d}{\sqrt{N}}\sigma K + \frac{d}{\sqrt{N}} K^\dagger \sigma + \frac{d^2}{N}\sigma^2 + \frac{2d}{\sqrt{N}}\alpha \tag{82}$$

and the effective action $\mathcal{A}_{ef}$ has the following power series expansion

$$\mathcal{A}_{ef} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}\frac{N}{2} \mathrm{Tr}\left[(K^\dagger K)^{-1}\left(\frac{d}{\sqrt{N}}\sigma K + \frac{d}{\sqrt{N}} K^\dagger\sigma + \frac{d^2}{N}\sigma^2 + \frac{2d}{\sqrt{N}}\alpha\right)\right]^{k+1}$$
$$- \frac{\sqrt{N}}{2f}\int d^n x dt\, \alpha(x,t) = \sqrt{N}\mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \cdots \tag{83}$$

Stability at large N implies that $\mathcal{A}^{(1)}$ must vanish. This condition gives

$$\int \frac{d^n k\, d\omega}{(2\pi)^{n+1}} \frac{d}{\omega^2 + \frac{d^2}{4}(k^2+m^2)^2} - \frac{1}{2f} = 0 \tag{84}$$

which is rapidly recognized as the usual mass gap formula,

$$\int \frac{d^n k}{(2\pi)^n}\frac{1}{k^2+m^2} - \frac{1}{2f} = 0 \tag{85}$$

At $n = 2$, defining a renormalized coupling constant f_R by

$$\frac{1}{f_R(\mu)} = \frac{1}{f} + \frac{1}{2\pi}\ln\frac{\Lambda^2}{\mu^2}, \tag{86}$$

where Λ is a Pauli Villars regularization mass and μ plays the role of the renormalization point, we get the well known formula

$$m^2 = \mu^2 e^{-\frac{2\pi}{f_R}} \tag{87}$$

$\mathcal{A}^{(2)}$ which furnishes the propagators is given by

$$\mathcal{A}^{(2)} = \int d^n x\, dt \int d^n x'\, dt' \Big\{\frac{1}{2}\sigma(x,t)\Gamma_{\sigma\sigma}(x-x', t-t')\sigma(x',t')$$
$$+\frac{1}{2}\sigma(x,t)\Gamma_{\sigma\alpha}(x-x',t-t')\alpha(x',t') + \frac{1}{2}\alpha(x,t)\Gamma_{\alpha\sigma}(x-x',t-t')\sigma(x',t')$$
$$+\frac{1}{2}\alpha(x,t)\Gamma_{\alpha\alpha}(x-x',t-t')\alpha(x',t')\Big\} \tag{88}$$

where

$$\Gamma_{\sigma\sigma}(q,\Omega) = -\frac{d^2}{2}\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \frac{1}{-i\omega + \frac{d}{2}(k^2+m^2)} \frac{1}{-i(\omega-\Omega) + \frac{d}{2}[(k-q)^2+m^2]}$$
$$-\frac{d^2}{2}\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \frac{1}{i\omega + \frac{d}{2}(k^2+m^2)} \frac{1}{i(\omega-\Omega) + \frac{d}{2}[(k-q)^2+m^2]} = 0 \tag{89}$$

$$\Gamma_{\alpha\alpha}(q,\Omega) = -2\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \frac{d}{\omega^2 + \frac{d^2}{4}(k^2+m^2)^2} \frac{d}{(\omega-\Omega)^2 + \frac{d^2}{4}[(k-q)^2+m^2]^2} \tag{90}$$

$$\Gamma_{\alpha\sigma}(q,\Omega) = -2d\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \frac{1}{-i\omega + \frac{d}{2}(k^2+m^2)} \frac{d}{(\omega-\Omega)^2 + \frac{d^2}{4}[(k-q)^2+m^2]^2} \tag{91}$$

$$\Gamma_{\sigma\alpha}(q,\Omega) = \Gamma_{\alpha\sigma}(q,\Omega) \tag{92}$$

The static limit may be obtained by integrating over Ω,

$$F(q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\Omega\, F(q,\Omega) \tag{93}$$

giving the same results as in the usual, non stochastic, approach. In particular,

$$\Gamma_{\alpha\alpha}(q) = -2\int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2+m^2}\frac{1}{(k-q)^2+m^2} \tag{94}$$

Although more complicated the integrals in (90) and (91) can be explicitly calculated. To illustrate the general method we consider the expression for

$\Gamma_{\alpha\alpha}$. We use,

$$\frac{d}{\omega^2 + \frac{d^2}{4}(k^2+m^2)^2} = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \frac{e^{-\frac{d}{2}(k^2+m^2)|t|}}{k^2+m^2}$$

$$\frac{d}{(\omega-\Omega)^2 + \frac{d^2}{4}[(k-q)^2+m^2]^2} = \int_{-\infty}^{+\infty} dt'\, e^{i(\omega-\Omega)t'} \frac{e^{-\frac{d}{2}[(k-q)^2+m^2]|t'|}}{(k-q)^2+m^2} \quad (95)$$

so that, integrating in ω and then in t', we get

$$-4\int \frac{d^n k}{(2\pi)^n} \int_0^{\infty} dt\, \cos(\Omega t) \frac{e^{-\frac{d}{2}(k^2+m^2)t}}{k^2+m^2} \frac{e^{-\frac{d}{2}[(k-q)^2+m^2]t}}{(k-q)^2+m^2} \quad (96)$$

This expression can be further simplified introducing Feynman parametric integrals, giving

$$-d^2 \int \frac{d^n k}{(2\pi)^n} \int_0^{\infty} dt\, \cos(\Omega t) \int_t^{\infty} d\alpha\, e^{-\frac{d}{2}(k^2+m^2)\alpha} \int_t^{\infty} d\beta\, e^{-\frac{d}{2}[(k-q)^2+m^2]\beta} \quad (97)$$

In fact, the integration in k is Gaussian and can be done immediately. We get

$$-\frac{d^2}{(2\pi d)^{n/2}} \int_0^{\infty} dt\, \cos(\Omega t) \int_0^1 dx \int_0^{\infty} da\, \theta(a - t/x)\, \theta(a - t/(1-x))$$

$$\times \frac{e^{-\frac{d}{2}a[x(1-x)q^2+m^2]}}{a^{n/2-1}} \quad (98)$$

where we have made a change of variables, $\alpha = ax$, $\beta = a(1-x)$.After integrating in t we obtain

$$-\frac{d^2}{(2\pi d)^{n/2}} \frac{1}{\Omega} \int_0^{1/2} dx \int_0^{\infty} da\, \sin(\Omega a x)\, \frac{e^{-\frac{d}{2}a[x(1-x)q^2+m^2]}}{a^{n/2-1}} \quad (99)$$

In two dimensions, $n = 2$, there is a considerable simplification. In this case we obtain the final result

$$\Gamma_{\alpha\alpha}(q,\Omega) = -\frac{1}{2\pi}\frac{i}{\Omega}\Bigg\{ \frac{1}{\sqrt{A^2+4q^2m^2}} \ln\left[\frac{A - q^2 - \sqrt{A^2+4q^2m^2}}{A - q^2 + \sqrt{A^2+4q^2m^2}} \frac{A + \sqrt{A^2+4q^2m^2}}{A - \sqrt{A^2+4q^2m^2}}\right]$$
$$-\frac{1}{\sqrt{B^2+4q^2m^2}} \ln\left[\frac{B + q^2 + \sqrt{B^2+4q^2m^2}}{B + q^2 - \sqrt{B^2+4q^2m^2}} \frac{B - \sqrt{B^2+4q^2m^2}}{B + \sqrt{B^2+4q^2m^2}}\right]\Bigg\} \quad (100)$$

where

$$A \equiv q^2 + \frac{2\Omega}{d}i \quad (101)$$

$$B \equiv -q^2 + \frac{2\Omega}{d}i \quad (102)$$

4 Conclusions

In this work we have studied the stochastic quantization of the non linear sigma model in the context of the $1/N$ expansion. We have considered two approaches. The first approach was based on Langevin equations and, to have a well defined equilibrium limit in the linear approximation, a nonlocal bilinear term was added to the free Lagrangian. The same term was, of course, subtracted from the interacting Lagrangian. Although very complicated for finite times, the added term has a simpler form at the field theoretical limit. The second method of quantization, a functional integral approach, is more adequate to the discussion of the ultraviolet divergencies. In that case a BRST symmetry strongly restricts the form of the allowed counterterms so that we are able to prove that all divergencies in the physical sector can be eliminated by a renormalization prescription. It should be remarked that this result is a consequence not only of the BRST symmetry but also of Ward identities related to the geometric nature of field ϕ_a. There is another (anti) BRS symmetry, [12], which ensures that the static limit corresponds to the usual non linear sigma model. Using this symmetry, the model can be written in a explicitly supersymmetric form. In fact, defining the superfields

$$\Phi_a(x,t,\overline{\theta},\theta) = \phi_a(x,t) + \overline{\theta}\, C_a(x,t) + \overline{C}_a(x,t)\,\theta + \lambda_a(x,t)\,\overline{\theta}\theta \quad (103)$$

$$\Xi(x,t,\overline{\theta},\theta) = \frac{1}{\sqrt{N}}[\sigma(x,t) + \overline{\theta}\, c(x,t) + \overline{c}(x,t)\,\theta + \alpha(x,t)\,\overline{\theta}\theta] \quad (104)$$

we find that the action (58) can be written as

$$\mathcal{S} = \int d^n x\, dt\, d\theta\, d\overline{\theta} \left\{ -\Phi_a \overline{D} D\, \Phi_a + \left(\Phi_a^2 - \frac{N}{2f}\right)\Xi + \mathcal{L}[\Phi] \right\} \quad (105)$$

where the covariant derivatives D and $\overline{D}$ are given by

$$D \equiv \frac{\partial}{\partial\overline{\theta}} - \theta\frac{\partial}{\partial t} \quad (106)$$

$$\overline{D} \equiv \frac{\partial}{\partial\theta} \quad (107)$$

and for simplicity we have taken $d = 2$. Noting that

$$\delta\left(\Phi_a^2 - \frac{N}{2f}\right) = \int \mathcal{D}\Xi\, \mathrm{e}^{\int d^n x\, dt\, d\theta\, d\overline{\theta} \left(\Phi_a^2 - \frac{N}{2f}\right)\Xi}, \quad (108)$$

the generating functional becomes

$$Z_{\mathrm{SS}}[\mathcal{J}] = \int \mathcal{D}\Phi\, \delta\left(\Phi_a^2 - \frac{N}{2f}\right) \mathrm{e}^{-\mathcal{S}_{\mathrm{SS}} - \int d^n x\, dt\, d\theta\, d\overline{\theta}\, \mathcal{J}_a \Phi_a} \quad (109)$$

where

$$\mathcal{S}_{\mathrm{SS}} = \int d^n x\, dt\, d\theta\, d\overline{\theta} \left(-\Phi_a \overline{D} D\, \Phi_a + \mathcal{L}[\Phi]\right) \quad (110)$$

and

$$\mathcal{J}_a = J_a(x,t)\delta(\overline{\theta})\delta(\theta) \quad (111)$$

Formula (109) shows that the constraint has to be imposed supersymmetrically to the free stochastic dynamics. This explicit supersymmetry is essential

to recover via dimensional reduction the correct static limit, [15, 16]. It is very important also, as we saw, to prove the renormalizability of the model. By contrast, the functional

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \mathcal{D}\lambda\, \delta(\phi_a^2 - \frac{N}{2f}) \exp\Big\{ - \int d^n x dt \Big[-\lambda_a^2 + \lambda_a\Big(\dot{\phi}_a + (-\Box + m^2)\phi_a + \Big) - J\phi \Big]\Big\} \quad (112)$$

which also corresponds to a free stochastic dynamics but with a non supersymmetric constraint, and which could appear perfectly reasonably, does not enjoy the explicit supersymmetry and it is not clear if its equilibrium limit does exist or even if it is renormalizable.

As a last remark we observe that, although having the same static limit, the two approaches that we have considered are not equivalent for finite values of the fifth time. In particular the geometric constraint satisfied by the fields holds only at the static limit in the case of Langevin method whereas it is imposed at all times in the functional integral method.

References

[1] G. Parisi and Y. Wu, Sci. Sinica 24, 483, 1981.

[2] H. Damgaard and H. Hüffel, Phys. Rep. 152, 227, 1987.

[3] D. Zwanziger, Nucl. Phys. B192, 259, 1981.

[4] M. Namiki, I. Ohba, K. Okano and Y. Yamanaka, Prog. Theor. Phys. 69, 1580, 1983 .

[5] J. Alfaro and B. Sakita, Phys. Lett. 121B, 339, 1983
J. Alfaro, Phys. Lett 200B, 80, 1988.

[6] S. Coleman, $1/N$, in " Pointlike Structures Inside and Outside Hadrons" (Ed. Zichichi), Plenum, New York (1981).

[7] K. Okano and L. Schülke, Nucl. Phys. B295[FS21], 105, 1988 .

[8] J. C. Niel and J. Zinn-Justin, Nucl. Phys. B280[FS18], 355, 1987.

[9] E. Gozzi, Phys. Rev. D28, 1922, 1983 .

[10] Y. Nakano, Prog. Theor. Phys. 69, 361, 1983 .

[11] J. Zinn-Justin, Nucl. Phys. B275[FS17] , 135, 1986
R. Iengo and S. Pugnetti, Nucl. Phys. B300[FS22] , 128, 1988 .

[12] G. Parisi and N. Sourlas, Phys. Rev. Lett. 43, 744, 1979.

[13] J. Cardy, Phys. Lett. 139B, 180, 1984.

[14] A. Klein and J. Fernando Perez, Phys. Lett. 125B, 473, 1983.

[15] R. Kirschner, Phys. Lett. 139B, 180, 1984.

[16] E. Gozzi Phys. Lett. 143B, 183, 1984.

[17] W. Grimus and H. Hüffel, Z. Phys. C18, 129, 1983 .

[18] J. H. Lowenstein and E. R. Speer, Nucl. Phys. B158, 397, 1979 .

[19] I. Ya. Aref'eva, Teor. Mat. Fiz. 36, 24, 1978 .

[20] A. Muñoz Sudupe and R. F. Alvarez–Estrada, J. Phys. A16, 3049, 1983 .

FIGURE CAPTIONS

Figure 1: Graphical notation used in the solution of Langevin equations.

Figure 2: Cancellation of contributions from one loop diagrams and bilinear σ insertions.

Figure 3: Feynman rules for the action (58), used in the functional integral method. The propagators for the λ_a and α fields vanish in the leading $1/N$ order.

Figure 4: Various mechanisms of cancellation involving insertions of $\lambda_a\phi_a$, $\phi_a\phi_a$ and $\phi_a C_a$.

Figure 5: Typical contributions to Green functions containing a $\phi_a\phi_a$ insertion. In diagram (a) the lines emanating from the $\phi_a\phi_a$ vertex go to different vertices.

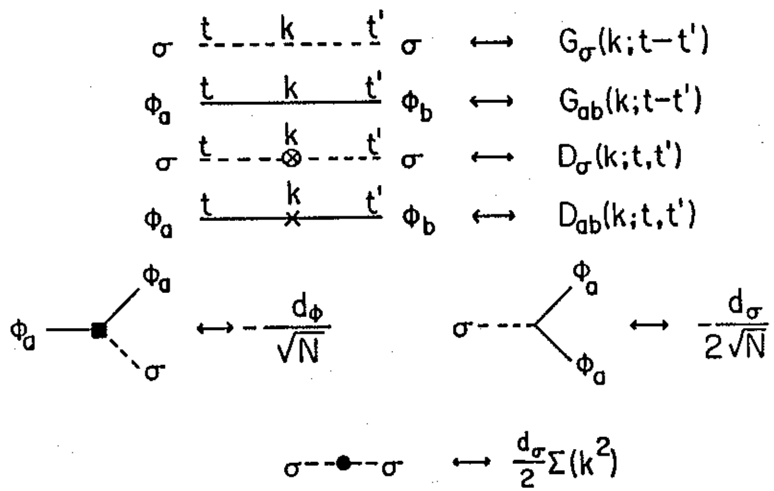

Figure 1

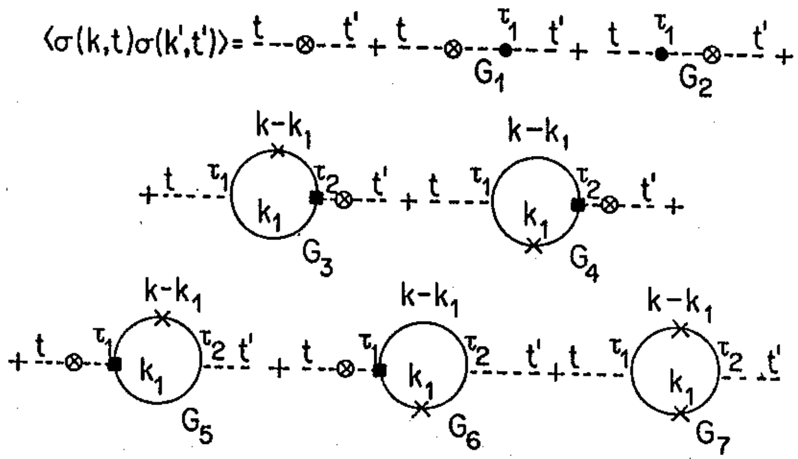

Figure 2

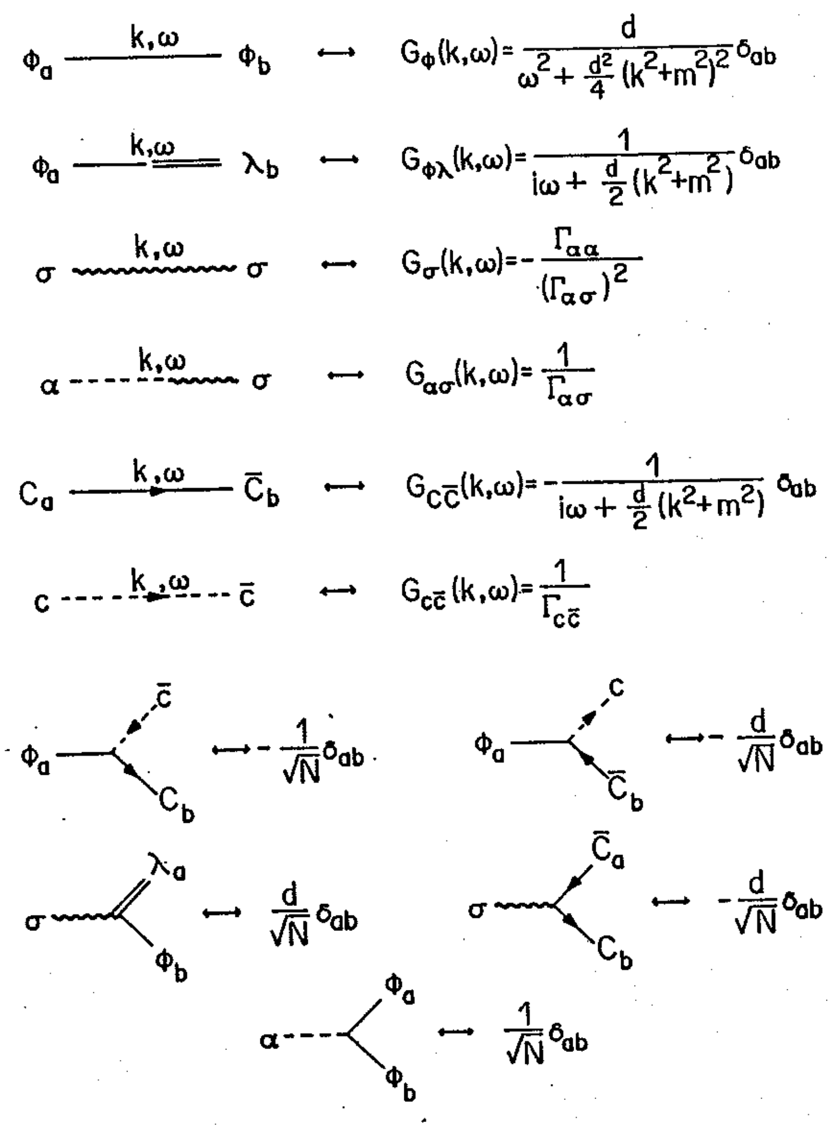

Figure 3

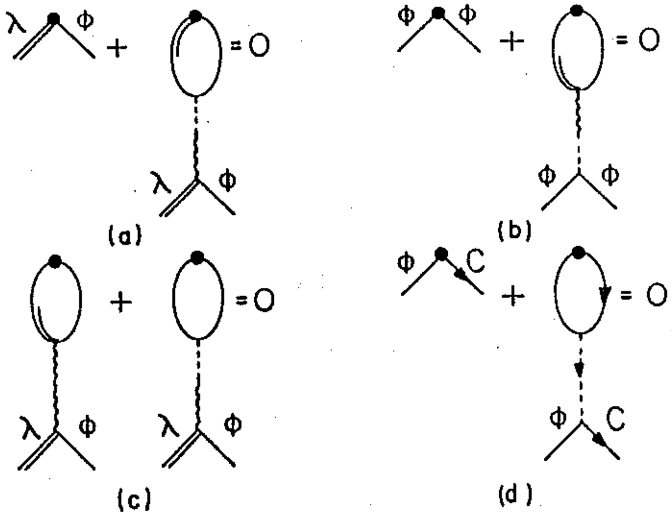

Figure 4

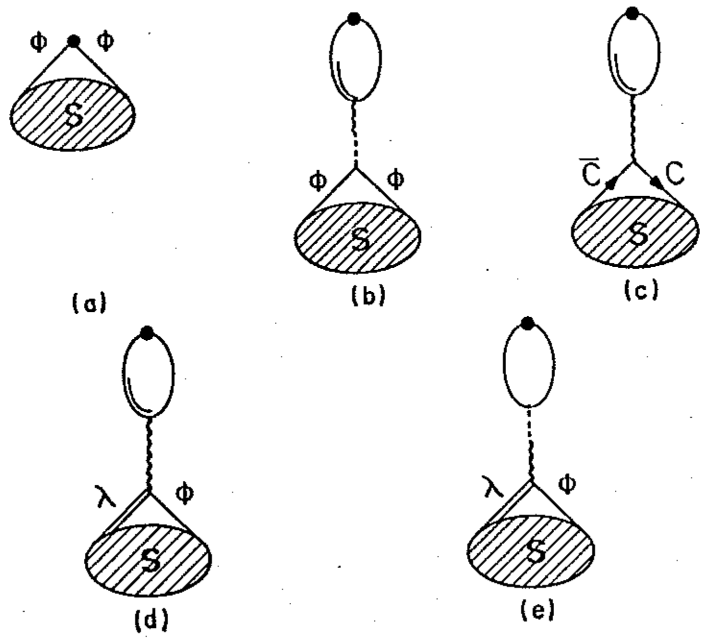

Figure 5