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CLOSED-FORM EXPRESSIONS FOR THERMAL GREEN FUNCTIONS IN FIELD THEORIES

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We study finite-temperature effects to one loop order in field theories, by relating them to the forward scattering of thermal particles. This approach allows for an exact evaluation of all temperature-dependent contributions to the thermal self-energy in terms of generalized Zeta functions. We obtain a closed-form expression for the two-point gluon function in thermal Yang-Mills theory.

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I. INTRODUCTION

There have been many studies of thermal Green functions in field theories [1–5], with special emphasis on the high-temperature domain. In particular, the contributions to the thermal two-point function, in the one-loop approximation, have been expressed as a series of terms in a high-temperature expansion.

One of the purposes of this work is to calculate in a closed and explicit form all finite-temperature effects associated with the two-point functions, which arise to one loop order in thermal field theories. We employ an analytic continuation of the imaginary time formalism and use an idea of Barton [6], which relates these functions to a momentum integral of the forward scattering amplitude of the thermal particles. This approach has been further developed and shown to be very useful for determining the partition functions in QCD and Quantum Gravity at high temperatures [7].

We will show that all finite-temperature effects arising in this case can be expressed in terms of the generalized Zeta function $\zeta(z, 1+q)$ for integer values of z, q being a ratio of the external momenta and temperature. This function can be represented as a series:

$$\zeta(z,1+q) = \sum_{n=1}^{\infty} \frac{1}{(q+n)^z} \qquad [Re\ z > 1] \qquad (1)$$

and reduces at $q = 0$ to Riemann's Zeta function $\zeta(z)$[8].

We begin section II by considering the massless $\lambda\phi^3$ theory in n-dimensions, as a model illustrating the main features of the method. In 6 dimensions, this model has many similarities with the Yang-Mills theory, such as a dimensionless coupling constant and asymptotic freedom. The results obtained in this case will be required in section III, where we study the Yang-Mills theory and present an exact expression for the gluon self-energy at finite temperatures.

II. THE SCALAR MODEL

In order to illustrate our approach and to derive several results which will be required later on, we consider here the massless $\lambda\phi^3$ model in n-dimensions. To order λ^2, the thermal part of the two-point function shown in Fig.(1a), is given by a momentum integral of the forward scattering amplitude of the thermal particle with momenta $Q_\mu = (|\mathbf{Q}|, \mathbf{Q})$[7], as represented in Figs. (1b) and (1c). We find, apart from temperature-independent terms, the following contribution:

$$\Pi_n(k,T) = \frac{-1}{(2\pi)^{n-1}} \int \frac{d^{n-1}Q}{2Q} N\left(\frac{Q}{T}\right) \left[\frac{1}{k^2 + 2Q.k} + \frac{1}{k^2 - 2Q.k}\right] \qquad (2)$$

where $Q \equiv |\mathbf{Q}|$ and $N\left(\frac{Q}{T}\right)$ denotes the Bose distribution.

The real-time limit of the Green function can be obtained from the analytically continued imaginary-time result via the prescription $k_0 = (1 + i\varepsilon)K_0$, where $\varepsilon \to 0^+$ and K_0 is real[9]. The presence of the factor $i\varepsilon$ associated with the imaginary part of k_0 will always be understood, unless otherwise indicated.

In terms of $x \equiv cos\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{Q}$, we find that the above expression becomes:

$$\Pi_n(k,T) = \frac{k^2}{2^n \pi^{\frac{n}{2}} \Gamma[(n-2)/2]} \int_{-1}^{1} \frac{(1-x^2)^{\frac{n-4}{2}}}{(k_0 - |\mathbf{k}|x)^2} dx \int_0^\infty Q^{n-3} N\left(\frac{Q}{T}\right) \frac{dQ}{Q^2 - [\frac{k^2}{2(k_0 - |\mathbf{k}|x)}]^2} \qquad (3)$$

For even values of $n > 2$, the Q integration can be expressed via elementary functions and in terms of:

$$J(y) = \int_0^\infty \frac{QdQ}{Q^2 + (2\pi T y)^2} \frac{1}{exp(Q/T) - 1} = \frac{1}{2}\theta[Re(y)]f(y) + (y \leftrightarrow -y) \qquad (4)$$

where $y \equiv \frac{i}{4\pi T}\frac{k^2}{k_0 - |\mathbf{k}|x}$,

$$f(y) = ln(y) - \frac{1}{2y} - \psi(y) \qquad (5)$$

Here $\psi(y) \equiv \frac{d}{dy} ln\Gamma(y)$ denotes the psi function[8].

We will start by considering the 4-dimensional case, and proceed with the x integration. Making a change of variables from x to y and using that $Re(y) = -\varepsilon' Re(k_0)$, where $\varepsilon' \to 0^+$, we find:

$$\Pi_4(k,T) = \theta[Re(-k_0)]\{F_4[q(k_0)] - F_4[-q(-k_0)]\} + (k_0 \leftrightarrow -k_0) \qquad (6)$$

where

$$q(k_0) \equiv i\frac{k_0 + |\mathbf{k}|}{4\pi T} \qquad (7)$$

$$F_4(q) = \frac{-iT}{8\pi|\mathbf{k}|}\int_C^q f(y)dy = \frac{iT}{8\pi|\mathbf{k}|}\left[ln\Gamma(q) - q\, ln(q) + \frac{1}{2}ln(q) + q + \text{constant}\right] \qquad (8)$$

The choice of C is immaterial, since the q-independent constant cancels out in expression (6)

Next we turn to the 6-dimensional $\lambda\phi^3$ model. From (3), we find with the help of eqs.(4,5) that:

$$\Pi_6(k,T) = \frac{T^2 k^2}{64\pi}\int_{-1}^{1}\frac{dx(1-x^2)}{(k_0 - |\mathbf{k}|x)^2}\left\{\frac{1}{6} - 2y^2\left[\theta[Re(y)]f(y) + (y \leftrightarrow -y)\right]\right\} \qquad (9)$$

The first term in the bracket yields a contribution proportional to T^2, which is leading in the high-temperature domain:

$$\Pi_6'(k,T) = \frac{1}{32\pi}\frac{T^2}{3}\left(1 - \frac{k_0^2}{|\mathbf{k}|^2}\right)\left[1 - \frac{k_0}{2|\mathbf{k}|}ln\left(\frac{k_0 + |\mathbf{k}|}{k_0 - |\mathbf{k}|}\right)\right] \qquad (10)$$

The real part of (10) is obtained by taking the absolute value of the expression which appears in the logarithm. From its argument, we get in the limit $\varepsilon \to 0^+$ the imaginary part:

$$Im\{\Pi_6'(k,T)\} = \frac{T^2}{192}\frac{|k_0 k^2|}{|\mathbf{k}|^3}\theta(-k^2) \qquad (11)$$

In the second term of (9) most of the factors can be easily integrated after changing variables from x to y. The most difficult part involves the integration over the psi function $\psi(y)$ multiplied by a power of y. The relevant integrals, discussed in the appendix A give:

$$I_1(q) = \int_C^q y f(y) dy = -q\zeta'(0, 1+q) + \zeta'(-1, 1+q) - \frac{3}{4}q^2 + \frac{q^2}{2} ln(q) + \text{constant} \tag{12}$$

where $\zeta'(z, 1+q)$ denotes the derivative with respect to z of the generalized Zeta function. Furthermore we find:

$$I_2(q) = \int_C^q y^2 f(y) dy = -q^2\zeta'(0, 1+q) + 2q\zeta'(-1, 1+q) - \zeta'(-2, 1+q) - \\ -\frac{11}{18}q^3 + \frac{q}{12} + \frac{q^3}{3} ln(q) + \text{constant} \tag{13}$$

In terms of these functions, we can then express the second part of Π_6, in a form which is similar to Π_4. After a straightforward calculation we obtain:

$$\Pi_6''(k, T) = \theta[Re(-k_0)] \{F_6[q(k_0)] - F_6[-q(-k_0)]\} + (k_0 \leftrightarrow -k_0) \tag{14}$$

where $q(k_0)$ is defined by (7) and

$$F_6(q) = -\frac{k^4}{16\pi|\mathbf{k}|^2} F_4 - \frac{T^2k^2}{8|\mathbf{k}|^3} \left[\frac{k_0}{2\pi} I_1(q) + iT I_2(q)\right] \tag{15}$$

with F_4 given by eq.(8).

The results obtained for Π_4 and Π_6 are precisely the one required in the case of the Yang-Mills theory, to which we now turn.

III. THE YANG-MILLS THEORY

The diagrams contributing to th: two-point thermal gluon function, via the forward scattering amplitude are shown in Fig.2. We are working in the Feynman gauge and for this reason it is necessary to consider also the contributions associated with the forward scattering of ghost particles, as indicated in Fig.(2b).

To order g^2, apart from an overall colour factor $N\delta^{ab}$, these graphs yield the following temperature-dependent contribution:

$$\Pi_{\mu\nu}(k, T) = \frac{-1}{(2\pi)^3} \int \frac{d^3Q}{2Q} N\left(\frac{Q}{T}\right) \times \\ \times \left[\frac{(2k^2 - 4k.Q)g_{\mu\nu} + 4Q_\mu Q_\nu + 2(Q_\mu k_\nu + Q_\nu k_\mu) - k_\mu k_\nu}{k^2 + 2k.Q} + (k \leftrightarrow -k)\right] \tag{16}$$

As it is well known[3], at finite temperatures both the transverse and the longitudinal parts of $\Pi_{\mu\nu}$ are independent, and can be expressed in terms of Π^μ_μ and Π_{00}. Using (16) we find that these components can be related to Π_4 and Π_6 as follows:

$$\Pi^\mu_\mu = \frac{T^2}{3} + 10k^2\Pi_4 \tag{17}$$

and

$$\Pi_{00} = \frac{3k_0^2 - 7|\mathbf{k}|^2}{2}\Pi_4 - 32\pi\frac{|\mathbf{k}|^2}{k^2}\Pi_6 \tag{18}$$

Thus, the study of the properties of the thermal two-point gluon function can be reduced to the analysis of the scalar factors which we have previously discussed. As we have seen, the present approach yields explicit expressions for Π_4 (eq. 6) and for Π_6 (eq. 10,14), giving the real and the imaginary parts of the Green functions. We have shown that these structures can be expressed via the factors F_4 (eq. 8) and F_6 (eq. 15) which are given in closed-form in terms of $ln\Gamma(q)$, $\zeta'(-1, 1+q)$ and $\zeta'(-2, 1+q)$, apart from other elementary functions. Since $ln\Gamma(1+q)$ equals to $\zeta'(0, 1+q)$ up to a constant (eq. A4), we see that the behaviour of the thermal self-energy can be described in terms of the derivatives of the generalized Zeta function $\zeta'(-n, 1+q)$ for $n = 0, 1, 2$.

We may now consider a special case of particular interest, corresponding to the high-temperature expansion of the thermal two-point functions. To this end we discuss in the first part of appendix B the asymptotic behaviour of $\zeta'(-n, 1+q)$ for small values of the parameter $q(k_0) = i\frac{k_0+|\mathbf{k}|}{4\pi T}$. Apart from irrelevant constant terms which cancel in eqs. (6) and (14) we find with help of (B4) that:

$$F_4(q) = \frac{Ti}{8\pi|\mathbf{k}|}\left[-\frac{1}{2}ln(q) + q(1-\gamma) - q\ ln(q) + \sum_{l=2}^{\infty}(-1)^l\frac{\zeta(l)}{l}q^l\right] \tag{19}$$

where γ denotes the Euler constant and

$$\begin{aligned} F_6(q) = \frac{iTk^2}{8|\mathbf{k}|^3}\Bigg\{&\left[\frac{iTk_0}{4\pi} - k^2\frac{1-\gamma}{(4\pi)^2}\right]q + \left[\frac{iTk_0}{8\pi}(2\gamma-1) - \frac{T^2}{4}\right]q^2 + \\ &+\frac{T^2}{9}(1-3\gamma)q^3 + \left[k^2\frac{1}{(4\pi)^2}\left(q-\frac{1}{2}\right) + \frac{iTk_0}{4\pi}q^2 - \frac{T^2q^3}{3}\right]ln(q) + \\ &+\sum_{l=2}^{\infty}(-1)^l\zeta(l)q^l\left[-\frac{k^2}{(4\pi)^2}\frac{1}{l} - \frac{iTk_0}{2\pi}\frac{q}{l+1} + \frac{T^2q^2}{l+2}\right]\Bigg\} \end{aligned} \tag{20}$$

We remark that these expressions yield, at most, linear contributions in T, which are non-leading in the high-temperature limit.

In order to compare our results with those of previous calculations, we now consider the real parts of the gluon Green functions Π^{μ}_{μ} (eq. 17) and Π_{00} (eq. 18). To this end, from the structures given for Π_4 (eq. 6) and Π''_6 (eq. 14) we obtain that

$$Re\{\Pi_n\} = Re\{F_n[q(k_0)] - F_n[-q(-k_0)]\} \tag{21}$$

where Π_n denotes Π_4 and Π''_6, respectively for $n = 4$ and $n = 6$.

From equations (10),(19),(20) and using the relation (21), we find that the real part of the gluon self-energy is given by an expression which is in agreement with those previously derived[2, 5] in the high temperature limit.

Another special case of some interest corresponds to the low-temperature limit of the thermal two-point functions. With the help of the relation (B8) describing the asymptotic behaviour of $\zeta'(-n, 1+q)$ for large q, we find that:

$$F_4(q) = \frac{iT}{8\pi|\mathbf{k}|}\sum_{l=1}^{\infty}\frac{B_{2l}}{2l(2l-1)}\frac{1}{q^{2l-1}} \tag{22}$$

where B_{2l} are the Bernoulli numbers, and

$$\begin{aligned} F_6(q) = \frac{T^2k^2i}{8|\mathbf{k}|^3}\Bigg\{&\frac{ik_0}{24\pi}ln(q) + \frac{k^2}{12T}\frac{1}{(4\pi)^2}\frac{1}{q} + \\ &+\sum_{l=2}^{\infty}\frac{B_{2l}}{2l}\frac{1}{q^{2l-3}}\left[\frac{k^2}{T}\frac{1}{(4\pi)^2}\frac{q^2}{2l+1} - \frac{ik_0}{2\pi}\frac{q}{2l+2} + \frac{T}{2k-3}\right]\Bigg\} \end{aligned} \tag{23}$$

As expected, in the low temperature limit the thermal part of the Green functions is small and quadratic in T, the dominant contribution being given by the vacuum (T=0) part.

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APPENDIX A

Here we derive relations (12,13). To this end we use the formula:

$$\zeta(z,y) = \frac{1}{1-z}\frac{d}{dy}\zeta(z-1,y) \qquad \text{(A1)}$$

which can be easily verified from eq. (1) Differentiating (A1) with respect to z, and integrating the result with respect to y gives the indefinite integral:

$$\int \zeta'(z,y)dy = \frac{1}{(1-z)^2}\zeta(z-1,y) + \frac{1}{1-z}\zeta'(z-1,y) \qquad \text{(A2)}$$

To prove (12) we require

$$\int y\psi(y)dy = y\, ln\Gamma(y) - \int ln\Gamma(y)dy \qquad \text{(A3)}$$

With the help of the relation[8]:

$$\zeta'(0,y) = ln\Gamma(y) - ln\sqrt{2\pi} \qquad \text{(A4)}$$

and making $z=0$ in (A2) we obtain that:

$$\int y\psi(y)dy = y[ln\Gamma(y) - ln\sqrt{2\pi}] - \zeta(-1,y) - \zeta'(-1,y) \qquad \text{(A5)}$$

Finally, using the functional relation[8]:

$$\zeta(-n,y) = -\frac{B_{n+1}(y)}{n+1} \qquad \text{(A6)}$$

where $B_{n+1}(y)$ denote the Bernoulli polynomials, we find the result:

$$\int y\psi(y)dy = \frac{1}{2}\left(y^2 - y + \frac{1}{6}\right) + y[ln\Gamma(y) - ln\sqrt{2\pi}] - \zeta'(-1,y) \qquad \text{(A7)}$$

Similarly, with help of the above relations we get:

$$\int y^2\psi(y)dy = \frac{1}{2}y^3 - \frac{1}{4}y^2 - \frac{1}{12}y + y^2[ln\Gamma(y) - ln\sqrt{2\pi}] - 2y\zeta'(-1,y) + \zeta'(-2,y) \qquad \text{(A8)}$$

The relations (12) and (13) can now be easily deduced from (A7) and (A8).

APPENDIX B

Here we discuss the behaviour of the generalized Zeta function for asymptotic values of the parameter $q = i\frac{k_0+|\mathbf{k}|}{4\pi T}$. To this end, we start from the representation[8]:

$$\zeta(z,1+q) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}e^{-qt}}{e^t-1}dt \qquad \text{(B1)}$$

At high temperatures, we can expand (B1) in a power serie of q. Making use of the integral representation of Riemann's Zeta function we find:

$$\zeta(z,1+q) = \sum_{l=0}^{\infty}\frac{\Gamma(z+l)}{\Gamma(z)}\frac{(-q)^l}{l!}\zeta(z+l) \qquad \text{(B2)}$$

Taking the derivative of (B2) with respect to z, we obtain in terms of the psi function $\psi(z)$ that:

$$\zeta'(z,1+q) = \sum_{l=0}^{\infty}\frac{(-q)^l}{l!}\frac{\Gamma(z+l)}{\Gamma(z)}\{[\psi(z+l)-\psi(z)]\zeta(z+l) + \zeta'(z+l)\} \qquad \text{(B3)}$$

We are actually interested in the values of $\zeta'(z,1+q)$ for $z \to -n$, where n is a natural number. After a long calculation we obtain that:

$$\begin{aligned}\zeta'(-n,1+q) = \sum_{l=0}^{n}\frac{q^l}{l!}\frac{n!}{(n-l)!}\left\{\zeta'(l-n) - \zeta(l-n)\sum_{k=n-l+1}^{n}\frac{1}{k}\right\} - \\ -\frac{q^{n+1}}{n+1}\left\{\gamma - \sum_{k=1}^{n}\frac{1}{k}\right\} + \\ + \sum_{l=n+2}^{\infty}(-1)^{n+l}q^l\frac{n!(l-n-1)!}{l!}\zeta(l-n)\end{aligned} \qquad \text{(B4)}$$

Using this relation, it is straightforward to arrive at the results given by eqs. (19) and (20).

In order to determine the behaviour of $\zeta(z,1+q)$ for asymptotically large values of q, it is convinient to make the change of variables $t \to \frac{t}{q}$ in (B1). Then, after deforming analitically the contour of integration to the real axis, we obtain the representation:

$$\zeta(z,1+q)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{q^z}\frac{t^{z-1}e^{-t}}{e^{\frac{t}{q}}-1} \tag{B5}$$

With help of the expansion:

$$\frac{x}{e^x-1}=\sum_{l=0}^{\infty}B_l\frac{x^l}{l!} \tag{B6}$$

where B_l are the Bernoulli numbers and using the integral representation of gamma functions[8] we find for large values of q the series:

$$\zeta(z,1+q)=\sum_{l=0}^{\infty}\frac{B_l}{l!q^{z+l-1}}\frac{\Gamma(z+l-1)}{\Gamma(z)} \tag{B7}$$

We next consider the derivative $\zeta'(z,1+q)$ for negative integer values of z. Proceeding as in the previous case, we arrive at the result:

$$\begin{aligned}\zeta'(-n,1+q)&=\left[\frac{B_{n+1}(q)}{n+1}+q^n\right]ln(q)-\frac{q^{(}n+1)}{(n+1)^2}\\&+\sum_{l=0}^{n+1}\frac{B_l}{l!}q^{n-l+1}(-1)^l\frac{n!}{(n-l+1)!}\sum_{k=n-l+2}^{n}\frac{1}{k}+\\&+\sum_{l=n+2}^{\infty}(-1)^n\frac{n!(l-n-2)!}{l!}\frac{B_l}{q^{l-n-1}}\end{aligned} \tag{B8}$$

By evaluating (B8) at $n=0,1,2$,we may now verify equations (22) and (23) in a straightforward way.

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FIGURES

FIG. 1. One loop diagram contributing to the thermal two-point function in the $\lambda\phi^3$ theory (a) and the corresponding forward scattering amplitude (b,c). Solid lines represent scalar particles.

FIG. 2. Forward scattering diagrams contributing to the thermal self-energy function in the Yang-Mills theory. Wavy lines denote gluons and broken lines represent ghost particles. Crossed graphs ($k \leftrightarrow -k$) are to be understood.

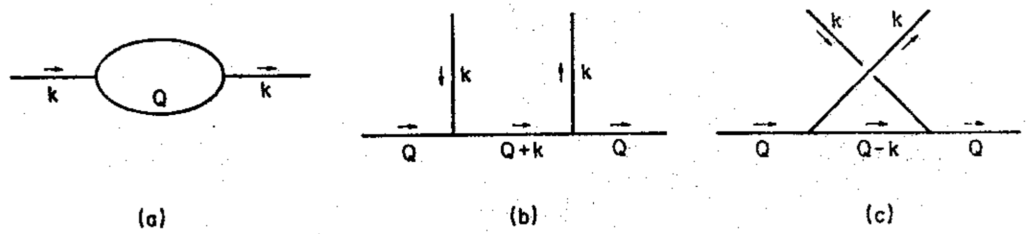

Fig. 1

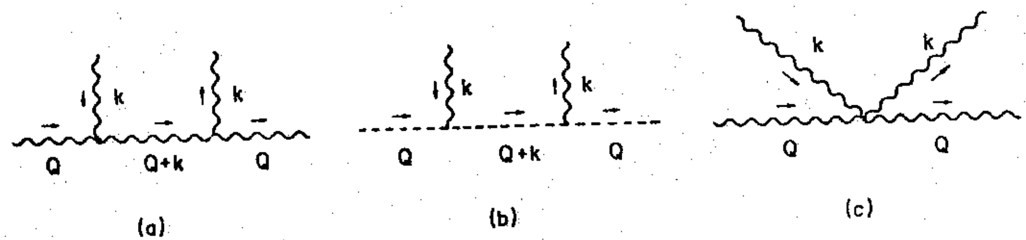

Fig. 2