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GAUGE INDEPENDENT ANALYSIS OF CHERN-SIMONS THEORY WITH MATTER COUPLING

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Gauge independent analysis of Chern-Simons theory with matter coupling.

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Abstract

We show that the Chern-Simons Theory coupled to complex scalars can be consistently quantised in the Hamiltonian formalism without gauge constraints. A new structure of the anyon operator displaying fractional spin and statistics follows logically from our analysis without any ad-hoc assumptions.

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The explicit construction of anyon operators exhibiting fractional spin and statistics in a canonical framework in 2+1 dimensional quantum field theory has remained controversial and debatable.

Work in this direction was pioneered by Semenoff[(1)] whose construction involved formal manipulations leading to some controversy and criticisms.[(2-5)] Ideas[(6)] akin to Semenoff's[(1)] have also been considered but subject to the same criticisms.[(2-5)] Besides the usual criticisms we emphasize that till now only gauge fixed Hamiltonian methods have been employed to discuss fractional spin and statistics of gauge dependent objects (the anyon operators). It is uncertain, therefore, whether the observed effects are physical or mere gauge artifacts. Indeed sometimes different results with different gauge fixing have been reported.[(7)]

In this letter we circumvent all these problems by showing, for the first time, that the 2+1 dimensional Chern-Simons (C.S.) theory coupled to complex scalars can be consistently quantised in the canonical formalism without any gauge fixing. All the space-time symmetries of the theory are preserved and the full Poincare algebra holds.

Our analysis naturally leads to the construction of gauge independent multivalued operators which create the physical states of the theory with arbitrary spin. We associate these operators with the anyon operators of the model. The structure of the anyon operator is completely new and, being gauge independent, the observed effects are physical. The anyon operators obey graded commutation relations which are compatible with the usual spin statistics theorem valid for fermions and bosons. Finally we show that the effect of the anyonic operators is to eliminate the gauge interactions from the Hamiltonian. Formal manipulations are avoided at all stages of the computations.

The Lagrangian of our model is given by,

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

where,

$$D_\mu = \partial_\mu + i A_\mu$$

with,

$$\varepsilon^{012} = 1 \quad , \quad g_{\mu\nu} = (+1, -1, -1)$$

It is invariant (upto a total divergence) under the gauge transformations,

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x)$$

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x) \tag{1}$$

The canonical momenta are,

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \quad , \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = \frac{\theta}{4\pi^2} \varepsilon_{ij} A^j$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (D_0 \phi)^* \quad , \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = D_0 \phi$$

so that , according to Dirac's[8] classification, the primary constraints are,

$$P_0 = \pi_0 \approx 0$$

$$P_i = \pi_i - \frac{\theta}{4\pi^2} \varepsilon_{ij} A^j \approx 0 \tag{2}$$

and the symbol $\approx$ stands for weak equality. The canonical Hamiltonian density is obtained from the Lagrangian via a Legendre transformation,

$$\mathcal{H}_c = \sum_{\chi = \text{Fields}} \pi_\alpha \dot{\chi}_\alpha - \mathcal{L}$$

$$= \pi^* \pi - A_0 J_0 - (D_i \phi)^* (D^i \phi) - \frac{\theta}{2\pi^2} \varepsilon^{ij} A_0 \partial_i A_j$$

where,

$$J_\mu = i \left[(D_\mu \phi)^* \phi - \phi^* (D_\mu \phi) \right] \tag{3}$$

is the conserved current.

The primary Hamiltonian is given by

$$H_p = \int d^2x \left(\mathcal{H}_c + u_0 \pi_0 + u_i P_i \right)$$

where u_0, u_i are arbitrary multipliers. Conserving the primary constraints with Hp and using the fundamental Poisson brackets (P.B.),

$$\{ A_\mu(x), \pi^\nu(y) \} = g_\mu^\nu \delta^{(2)}(x-y)$$

$$\{ \phi(x), \pi(y) \} = \{ \phi^*(x), \pi^*(y) \} = \delta^{(2)}(x-y)$$

yields the secondary constraint,

$$S = J_0 + \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i A_j \approx 0$$

No further constraints are generated via this iterative procedure. We find that P_0 is first class while P_i and S are second class constraints. It is, however, essential to extract the maximal set of first class constraints.[9] The following combination of the second class constraints,

$$P = \partial^i P_i + S$$
$$= \partial^i \pi_i + J_0 + \frac{\theta}{4\pi^2} \varepsilon^{ij} \partial_i A_j \approx 0$$

is first class.

The maximal set of first class constraints is thus given by P_0 and P while P_i are second class. This completes our classification of constraints.

We now construct the first class Hamiltonian from its canonical expression,

$$\mathcal{H}_F = \mathcal{H}_c - \int dx\,dy \, \{\mathcal{H}_c , P_i(x)\} \, P^{-1}_{ij}(x,y) \, P_j(y)$$

where,

$$P^{-1}_{ij}(x,y) = \frac{2\pi^2}{\theta} \varepsilon_{ij} \, \delta(x-y)$$

is the inverse of the matrix of the P.B's among P_i and P_j. We obtain,

$$\mathcal{H}_F = \pi^* \pi - (D_i\phi)^* (D^i\phi) + \frac{2\pi^2}{\theta} \varepsilon_{ij} J^i P^j - A_0 P$$

To $\mathcal{H}_F$ we can add linear combinations of the two first class constraints so that the total first class Hamiltonian is,

$$\mathcal{H}_T = \mathcal{H}_F + u\,\pi_0 + v\,P$$

where u,v are arbitrary multipliers reflecting the gauge invariances in the theory associated with the first class constraints. It is possible to fix the gauge so that this arbitrariness is eliminated. This is the usual course followed in the literature.[1,6,7]

An alternative approach,[10] however, is to determine u and v so that the correct Heisenberg's equations of motion are reproduced. The first step is to compute the Dirac brackets (D.B.) among the fundamental fields and their momenta, generically denoted by χ , defined as,[8]

$$\{\chi(x), \chi(y)\}_{D.B.} = \{\chi(x), \chi(y)\}_{P.B.} - \int dz\,dz' \{\chi(x), P_i(z)\}_{PB} P^{-1}_{ij}(z,z') \{P_j(z'), \chi($$

The ones which differ from their P.B's are given by,

$$\{A_i(x), A_j(y)\}_{DB} = \frac{2\pi^2}{\theta} \varepsilon_{ij} \, \delta(x-y)$$

$$\{A_i(x), \pi_j(y)\}_{DB} = \frac{1}{2} g_{ij} \, \delta(x-y)$$

$$\{\pi_i(x), \pi_j(y)\}_{DB} = \frac{\theta}{8\pi^2} \varepsilon_{ij} \, \delta(x-y) \qquad (4$$

which are compatible with setting the second class constraint $P_{i(2)}$ strongly equal to zero. The correct equations of motion,

$$\{ \chi , \int \mathcal{H}_T \}_{DB} = \partial \chi$$

calculated via the Dirac brackets(4) can now be reproduced with the unique choice,

$$u = \partial_0 A_0 \quad , \quad v = 0$$

The same analysis is next repeated for the momentum operator M_i defined via the canonical energy momentum tensor,

$$M_i^c = \theta_{0i}^c$$

where,

$$\theta_{\mu\nu}^c = \sum_{\varphi = \phi, \phi^*, A_\mu} \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \partial_\nu \varphi - \mathcal{L}\, g_{\mu\nu}$$

$$= (D_\mu \phi)^* (\partial_\nu \phi) + (D_\mu \phi)(\partial_\nu \phi^*) + \frac{\theta}{4\pi^2} \varepsilon_{\sigma\mu\lambda} A^\sigma \partial_\nu A^\lambda - \mathcal{L}\, g_{\mu\nu}$$

The final expressions for the generators of space-time translations may be written in a Lorentz covariant form,

$$\theta_{0\mu} = \theta_{0\mu}^c + u_{0\mu} \Pi_0 + v_{0\mu} P \tag{5}$$

with,

$$u_{0\mu} = \partial_\mu A_0 \quad , \quad v_{0\mu} = 0$$

so that,

$$\{ \chi , \int \theta_{0\mu} \}_{DB} = \partial_\mu \chi$$

It is simple to check that the constraints have vanishing D.B. with $\int \theta_{00}$ so that they are fixed in time. In the quantised version, therefore, $\theta_{0\mu}$ replaces $\theta_{0\mu}^c$ while the D.B. are transformed into commutators following the usual prescription $i\{ \, , \, \}_{DB} \to [\quad , \quad]$, and operator symmetrisation is implied whenever products of operators occur. The other space-time generators (i.e. rotations and boosts) can be treated in an identical fashion. It is found that the fields transform normally and there are no anomalies. Finally it is obligatory to check the Poincare algebra. From(5) and the D.B. (4), we find the relations,

$$\{ M^\mu , M^\nu \}_{DB} = 0$$

$$\{ M^\mu , L^{\lambda\nu} \}_{DB} = g^{\mu\nu} M^\lambda - g^{\mu\lambda} M^\nu$$

$$\{ L^{\mu\nu} , L^{\rho\sigma} \}_{DB} = g^{\mu\rho} L^{\nu\sigma} - g^{\nu\rho} L^{\mu\sigma} + g^{\nu\sigma} L^{\mu\rho} - g^{\mu\sigma} L^{\nu\rho}$$

valid on the physical states, which are annihilated by the first class constraints, and L is the angular momentum obtained from (5).Consequently the Poincare algebra is satisfied. This completes our discussion of the quantisation of the model without gauge constraints.

Next we try to obtain the operators which create the physical one-particle states of the model. The physical states are gauge invariant, and we can be assured of this property if the operatos creating these states from the vacuum are also gauge invariant. Moreover, following usual convention[5] we define the one particle states to be those states which carry one unit of the charge $Q = \int d^2 x\, J_0$ (eq. 3) i.e. states,

$$|1\rangle = \hat{\phi}|0\rangle$$

such that the creation operator $\hat{\phi}$ obeys,

$$[J_0(x), \hat{\phi}(y)] = \delta^{(2)}(x-y)\,\hat{\phi}(y) \tag{6}$$

It is simple to check that a general structure for $\hat{\phi}$ satisfying the above properties may be written,

$$\hat{\phi}(x) = e^{\int dy\,\Omega(x-y)\,J_0(y)}\; e^{i\int_{\infty}^{x} dy_i A_i(y)}\; \phi(x) \tag{7}$$

where Ω is, as yet, an undetermined function. The first thing to notice is that although the A_i's are non commuting (following from the D.B. (4)), they commute under the integrals i.e.

$$\int dy_i\, dz_j\, [A_i(y), A_j(z)] = 0$$

Moreover J_0 commutes with itself as well as with A_i, so that path ordering the exponentials is not required and the order in which the two exponetials appear is also unimportant. The operator $\hat{\phi}$ is invariant under the gauge transformations (1) while eq.(6) is implied because of the non trivial commutator,

$$[J_0(x), \phi(y)] = \delta^{(2)}(x-y)\,\phi(y) \tag{8}$$

following from the fundamental D.B. Generic one particle states may be obtained by a linear superposition of $\hat{\phi}$. Similarly one antiparticle states will be created by $\hat{\phi}^{\dagger}$.

In order to determine the function Ω (x-y) in (7), we first compute the general n-particle state functional,

$$|\Psi_n\rangle = \Big(\prod_{i=1}^{n} \hat{\phi}(x_i)\Big)|0\rangle \tag{9}$$

To simplify this, note that eq. (8) implies, by the Baker-Campbell-Hausdorff formula,

$$e^{\int dy\,\Omega(x-y)\,J_0(y)}\,\phi(z)\,e^{-\int dy\,\Omega(x-y)\,J_0(y)} = e^{\Omega(x-z)}\,\phi \tag{10}$$

Using this formula the n-particle state functional (9) may be expressed as,

$$|\Psi_n\rangle = \exp\Big[-\sum_{j=1}^{n}\sum_{i=1}^{j-1}\Omega(x_i-x_j)\Big]\Big\{\exp\Big[\sum_{i=1}^{n}\int dy\,\Omega(x_i-y)\,J \quad \prod_{i=1}^{n}\bar{\phi}(x_i)\,|0\rangle\Big\} \tag{11}$$

where,

$$\bar{\phi}(x) = \exp\Big(i\int^{x} dy_i\, A_i(y)\Big)\;\phi(x)$$

Now the general form of the n-particle state functional following from the representation theory of the braid group is given by,

$$\Psi_S[\chi(x_1)\cdots\chi(x_n);t] = \exp\Big[-2is\sum_{j=1}^{n}\sum_{i=1}^{j-1}\omega(x_i-x_j)\Big]\Psi_0[\chi(x_1)\cdots\chi(x_n); \tag{12}$$

where, it was shown by Forte and Joliceur[5] that for C.S. theory with matter coupling the generalised spin factor S is a function of the parame ter θ , henceforth denoted by $S(\theta)$. For the Klein-Gordon field $S(\theta)=\frac{1}{\theta}$. It may be observed, however, that the explicit functional form of $S(\theta)$ is immaterial for the ensuing analysis. $\Psi_0[\chi(x_1)\cdots\chi(x_n);t]$ is an n-particle state functional with Bose statistics and $\omega(x-y)$ is the multi valued polar angle of the vectors $x-y$,

$$\omega(x-y) = \arctan\frac{x^2-y^2}{x^1-y^1}$$

Going back to eq.(11) we observe that the expression in the curly bracket represents a gauge invariant functional with commuting one particle cocyles (because $[\int dy\,\Omega(x-y)J_0(y), \int dy'\,\Omega(x'-y')J_0(y')]=0$) and hence may be identified with Ψ_0 (12). The correspondence between the equations (11) and (12) becomes complete if one substitutes,

$$\Omega(x_i-x_j) = 2iS(\theta)\,\omega(x_i-x_j)$$

Hence the final expression for the one particle creation operator (7) is,

$$\hat{\phi}(x) = e^{2iS(\theta)\int dy\,\omega(x-y)J_0(y) + i\int^x dy_i A_i(y)}\,\phi(x) \tag{13}$$

which is multivalued due to the presence of $\omega(x-y)$. This is our anyonic field operator since it creates states (11) with arbitrary spin $S=S(\theta)$ when acting on the vacuum. It is different from all previous constructions suggested in the literature [1,6,7], being manifestly gauge invariant. This is significant because it may be recalled that the usual anyonic operators found in the literature [1,5-7], which are gauge dependent, are obtained in the Hamiltonian formalism with a specific gauge choice.

It is not clear, therefore, whether their anyonicity is a physical effect or an artifact of the gauge.

To study the statistics of $\hat{\phi}(x)$(13) we compute the product $\hat{\phi}(x)\hat{\phi}(y)$ and exploit the formula (10) to obtain,

$$\hat{\phi}(x)\,\hat{\phi}(y) = e^{\pm 2i\pi S(\theta)}\,\hat{\phi}(y)\,\hat{\phi}(x)$$

since $\omega(x-y) - \omega(y-x) = \pm\pi$.

The sign ambiguity in the phase arises because the function ω is only defined mod. (2π). Physically it reflects the arbitrariness present in the exchange of two particles which may be done either via a clockwese or an anticlockwise rotation. The above equation reveals that the fields obey graded commutation relations. For integral values of $S(\theta)$ (corresponding to bosons), commutators are obtained while half-integral values of $S(\theta)$ (corresponding to fermions) yield anticommutators. Thus the usual spin-statistics theorem valid for bosons and fermions is reproduced.

To further understand the implications of the construction (13), we consider the interaction piece of the Hamiltonian (5) and reexpress it in terms of the hat variables (13). We obtain

$$(D_i\phi)^*(D_i\phi) = \partial_i\left(e^{-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)}\hat{\phi}\right)^*\partial_i\left(e^{-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)}\right. \tag{14}$$

We observe from (14) that the explicit dependence on the interaction has been eliminated by the use of the hat variables. Thus we may consider our theory either in terms of single valued fields ϕ , having normal spin and statistics, in the presence of interactions or, alternatively, in term of multivalued fields $\hat{\phi}$, with generalised spin and statistics, which are effectively free.[1]

To conclude, this is the first paper which shows that the C.S. theory coupled to complex scalars can be systematically quantised in the canonical formalism without gauge constraints. All the space-time symmetries are preserved and the full Poincare algebra is valid. Our analysis naturally leads to the construction of multivalued anyonic operators which create physical states with arbitrary spin. These operators satisfy graded commutation relations which are compatible with the spin-statistics theorem. The anyonic operators found here are completely new and improve upon the previous constructions[1,6,7] since these operators are gauge independent and so the observed effects are physical.

The earlier papers,[1,2,6,7] however, employ specific gauge fixing techniques to discuss anyonicity of gauge dependent objects. Consequently their interpretation remains obscure. Moreover we have avoided the usual formal manipulations, which have led to controversies and criticisms,[2-5] in obtaining the anyonic operators. The anyonic operators effectively eliminate the interaction from the Hamiltonian thereby resulting in the dual interpretation of the theory, either in terms of single valued fields with usual spin-statistics having gauge interactions or in terms of multivalued fields with arbitrary spin-statistics having no gauge interactions, originally mooted by Semenoff.[1] The extension of our analysis for fermionic matter couplings and the effects of including a Maxwell term in the theory will be discussed elsewhere.

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