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QUANTISATION OF MATTER COUPLED CHERN SIMONS THEORY WITHOUT GAUGE CONSTRAINTS, AND THE ANYON OPERATOR

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Quantisation of matter coupled Chern Simons theory without gauge constraints, and the anyon operator

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Abstract

We show that the Chern-simons theory coupled to fermions can be consistently quantised in the Hamiltonian formalism without gauge constraints. A new structure of the anyon operator is obtained. The connection with the Lagrangian approach is illuminated.

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The study of 2+1 dimensional matter coupled Chern Simons (C.S) theory has recently generated considerable interest both from the theoretical and experimental points of view[1]. Although the quantum mechanics of these theories seem to be reasonably well understood[2], there are several criticisms and controversies regarding the field theoretical results[3-7]. The Lagrangian (path integral) formulation[3], for example, yields results which disagrees with the canonical Hamiltonian formalism[4,8,9]. Even within this canonical formalism, which will also be the subject of our analysis, contradictory results have been reported[4,8,9]. Moreover the construction of anyon operators displaying fractional spin and statistics involves ambiguous manipulations which have been criticised[3-7]. A possible clue to the conflicting results is the usage of gauge fixed Hamiltonian methods[4,8,9] to discuss the physics of gauge dependent (anyon) operators. It is not surprising, therefore, that different results with different gauge choices have been found in the literature[9].

Recently we[10] suggested an alternative, gauge independent Hamiltonian method of quantising the matter coupled C.S. theory. In this paper we use that formalism to show that the C.S. theory with fermionic matter coupling can be consistently quantised. All the space time symmetries of the theory are preserved and the complete Poincare algebra is satisfied. Our analysis leads to the construction of multivalued operators which create the physical states of the theory with arbitrary spin. These are, therefore, the anyon operators of the model. Contrary to earlier structures[4,6,8,9] the anyon operator here is gauge invariant so that the observed effects are physical and not gauge artifacts. The anyon operators obey graded commutation relations consistent with the spin-statistics theorem valid for bosons and fermions. The connection of our Hamiltonian analysis to the Lagrangian (path-integral)formulation[3] is elucidated. Finally we show that the redefined theory in terms of the anyon variables is effectively free. Formal manipulations, which earlier gave rise to controversies[3-8], have been avoided.

The Lagrangian of our model is given by:

$$\mathcal{L} = \frac{1}{2}\overline{\psi} i \overleftrightarrow{\not\partial}\psi + \overline{\psi}\not{A}\psi + \frac{\theta}{4\pi^2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \qquad (1)$$

where we use the form,

$$\overline{\psi}\overleftrightarrow{\not\partial}\psi = \overline{\psi}\overrightarrow{\not\partial}\psi - \overline{\psi}\overleftarrow{\not\partial}\psi \qquad (2)$$

to preserve hermiticity. Without any loss of generality the coupling has been set equal to unity while the γ matrices in 2+1 dimensions satisfy the algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad , \quad \gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho \qquad (3)$$

with, $g^{\mu\nu} = (+1, -1, -1)$ and $\epsilon^{012} = 1$.

The Lagrangian (1) is invariant (upto a total divergence) under the gauge transformations,

$$\begin{aligned} &\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \quad , \quad \overline{\psi}(x) \rightarrow \overline{\psi}(x)e^{-i\alpha(x)} \\ &A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\alpha(x) \end{aligned} \qquad (4)$$

The canonical momenta are given by,

$$\Pi_0 = \frac{\partial\mathcal{L}}{\partial \dot{A}^0} = 0 \quad , \quad \Pi_i = \frac{\partial\mathcal{L}}{\partial \dot{A}^i} = \frac{\theta}{4\pi^2}\epsilon_{ij}A^j$$
$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot\psi_\alpha} = -\frac{i}{2}(\overline{\psi}\gamma_0)_\alpha \quad , \quad \overline{\Pi}_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\overline{\psi}}_\alpha} = -\frac{i}{2}(\gamma_0\psi)_\alpha \quad , \tag{5}$$

so that Dirac's[11] classification gives the primary constraints,

$$P_0 = \Pi_0 \approx 0 \qquad , \qquad P_i = \Pi_i - \frac{\theta}{4\pi^2}\epsilon_{ij}A^j \approx 0$$
$$\eta_\alpha = \Pi_\alpha + \frac{i}{2}(\overline{\psi}\gamma_0)_\alpha \approx 0 \quad , \quad \overline{\eta}_\alpha = \overline{\Pi}_\alpha + \frac{i}{2}(\gamma_0\psi)_\alpha \approx 0 \tag{6}$$

and the symbol $\approx$ denotes weak equality.

The canonical Hamiltonian is obtained from the Lagrangian by a formal Legendre transformation,

$$\mathcal{H}^C = -\frac{i}{2}\overline{\psi}\gamma_k\overleftrightarrow{\partial}^k\psi - \overline{\psi}\gamma_k A^k\psi - A_0\left(J_0 + \frac{\theta}{2\pi^2}\epsilon^{ij}\partial_i A_j\right) \tag{7}$$

where J_μ is the conserved current,

$$J_\mu = \overline{\psi}\gamma_\mu\psi \tag{8}$$

The primary Hamiltonian is,

$$H^P = \int d^2x[\mathcal{H}^C + u_0P_0 + u_iP_i + \eta_\alpha c_\alpha + \overline{c}_\alpha\overline{\eta}_\alpha] \tag{9}$$

where u_0, u_i are ordinary multipliers and $c_\alpha, \overline{c}_\alpha$ are Grassman multipliers.

Conserving the primary constraints with H^P yields the secondary constraint,

$$S = J_0 + \frac{\theta}{2\pi^2}\epsilon^{ij}\partial_iA_j \approx 0 \tag{10}$$

which is Gauss' law. It may be checked that no further constraints are generated by this iterative proceedure.

We find that P_0 is a first class constraint while $P_i, \eta_\alpha, \overline{\eta}_\alpha$ and S are second class. In usual electrodynamics the Gauss constraint S appears as first class but here it is second class. This is because the maximal set of first class constraints has not been extracted[12]. It is simple to verify that the following combination of second class constraints,

$$P = \partial^iP_i + S + ie(\eta_\alpha\psi_\alpha + \overline{\psi}_\alpha\overline{\eta}_\alpha)$$
$$= \partial^i\Pi_i + J_0 + \frac{\theta}{4\pi^2}\epsilon^{ij}\partial_iA_j + ie(\eta_\alpha\psi_\alpha + \overline{\psi}_\alpha\overline{\eta}_\alpha) \tag{11}$$

is first class. Subsequently when we work with Dirac brackets (D.B.), the second class constraints can be strongly set equal to zero whence P reduces to the familiar Gauss

constraint S (eq.10). The maximal set of first class constraints is given by P_0 and P, while P_i, η_α and $\overline{\eta}_\alpha$ are second class. This completes the classification of constraints.

We next compute the D.B. in the usual way[11]. The ones which differ from their P.B. are listed below:

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\}_{D.B.} = -4\{\Pi_\alpha(x), \overline{\Pi}_\beta(y)\}_{D.B.} = -i(\gamma_0)_{\alpha\beta}\delta(x-y)$$
$$\{\psi_\alpha(x), \Pi_\beta(y)\}_{D.B.} = \{\overline{\psi}_\alpha(x), \overline{\Pi}_\beta(y)\}_{D.B.} = -\frac{1}{2}\delta_{\alpha\beta}\delta(x-y)$$
$$\{A^i(x), A^j(y)\}_{D.B.} = \left(\frac{4\pi^2}{\theta}\right)^2\{\Pi^i(x), \Pi^j(y)\}_{D.B.} = \frac{2\pi^2}{\theta}\epsilon^{ij}\delta(x-y) \tag{12}$$
$$\{A^i(x), \Pi^j(y)\}_{D.B.} = \frac{g^{ij}}{2}\delta(x-y)$$

which are consistent with setting the second class constraints strongly zero. The total Hamiltonian is given by,

$$\mathcal{H}^T = \mathcal{H}^C + u\Pi_0 + vP \quad , \tag{13}$$

where u and v are arbitrary multipliers reflecting the gauge invariances of the theory associated with the two first class constraints.

With this material we can discuss the quantisation of the model. There are two options. We may fix the gauge (by choosing two gauge constraints not following from the Lagrangian) so that the freedom in u and v is completely eliminated. This is the usual course adopted in the literature[4,8,9]. Alternatively[13] we may determine u and v so that the Heisenberg's equations are reproduced for the canonical variables.

It may be easily verified that the correct equations of motion,

$$\left\{\chi, \int\mathcal{H}^T\right\}_{D.B.} = \partial_0\chi \tag{14}$$

for all the canonical variables χ, calculated by using (13) and (12), can be obtained with the unique choice,

$$u = \partial_0A_0 \qquad v = 0 \tag{15}$$

for the arbitrary multipliers. The same analysis is now repeated for the momentum operator M_i defined from the canonical energy momentum tensor,

$$M_i{}^C = \int \theta_{0i}{}^C \tag{16}$$

where,

$$\theta^C_{\mu\nu} = \partial_\nu\psi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)} + \partial_\nu\overline{\psi}\frac{\partial\mathcal{L}}{\partial(\partial^\mu\overline{\psi})} + \frac{\partial\mathcal{L}}{\partial(\partial^\mu A_\lambda)}\partial_\nu A_\lambda - \mathcal{L}g_{\mu\nu}$$
$$= \frac{i}{2}\overline{\psi}\gamma_\mu\overleftrightarrow{\partial}_\nu\psi - \frac{\theta}{4\pi^2}\epsilon_{\mu\sigma\lambda}A^\sigma\partial_\nu A^\lambda - g_{\mu\nu}\mathcal{L} \quad . \tag{17}$$

The final expressions for the generators of space time translations may be expressed in a Lorentz covariant form,

$$\theta^T_{0\mu} = \theta^C_{0\mu} + u_{0\mu}\Pi_0 + v_{0\mu}P \tag{18a}$$

with,

$$u_{0\mu} = \partial_\mu A_0 \qquad v_{0\mu} = 0 \tag{18b}$$

whereby,

$$\left\{ \chi \quad , \int \theta^T_{0\mu} \right\}_{D.B} = \partial_\mu \chi \tag{19}$$

Thus in the quantised version of the theory without gauge constraints, the canonical energy momentum tensor $\theta^C_{0\mu}$ is replaced by $\theta^T_{0\mu}$ while the D.B.(12) are converted to the equal time (anti) commutators following the usual prescription,

$$\{P, Q\}_{D.B.} \rightarrow i(PQ - (1)^{n_P n_Q} QP) \tag{20}$$

where $n_P = 0(1)$ for bosonic (fermionic) P.

We can similarly deal with the other space time generators (i.e.rotations and boosts) of the theory. It is observed that the fields have their usual transformation properties without any anomalies. Finally we find, after an extensive algebra, that the generators of space time symmetries defined from the complete energy momentum tensor (18) fulfill the entire Poincare algebra via the D.B. (12). This completes our analysis of the quantisation of the model without gauge fixing.

To obtain the Fock space we make the following ansatz,

$$\hat{\psi}(x) = e^{\int dy \Omega(x-y) J_0(y) - i \int_\infty^x dy_i A_i(y)} \psi(x) \tag{21}$$

for the one particle creation operator where $\Omega(x-y)$ is, as yet, an undetermined function. Note that although the gauge potentials are non-commuting (see 12), these commute under the integrals i.e.,

$$\int dy_i dz_j [A_i(y), A_j(z)] = 0 \tag{22}$$

The other terms in the exponent in eq.(21) commute so that the exponential need not be path ordered. Moreover $\hat{\psi}(x)$ is invariant under the gauge transformations (4),thereby satisfying the essential prerequisite that it creates states which are gauge invariant. Finally it is easy to verify that the one particle states,

$$|\psi_1\rangle = \hat{\psi}(x)|0\rangle \tag{23}$$

indeed carry one unit of the charge $Q = \int d^2x J_0$ (eq.8) because,

$$[Q, \hat{\psi}(x)] = \hat{\psi}(x) \tag{24}$$

which follows from the non-vanishing commutator,

$$[J_0(x), \psi(y)] = \delta(x-y)\psi(y) \tag{25}$$

calculated by the D.B.(12).

In order to fix $\Omega(x-y)$ in (21), we compute the n-particle state functional obtained from (21),

$$|\psi_n\rangle = \left(\prod_{i=1}^n \hat{\psi}(x_i) \right) |0\rangle \tag{26}$$

This may be simplified by exploiting the relation,

$$e^{\int dy \Omega(x-y) J_0(y)} \psi(z) e^{-\int dy \Omega(x-y) J_0(y)} = e^{\Omega(x-z)} \psi(z) \tag{27}$$

which follows from (25) and the Baker-Campbell-Hausdorff formula. We obtain,

$$|\psi_n\rangle = \exp[-\sum_{j=1}^n \sum_{i=1}^{j-1} \Omega(x_i - x_j)] \left\{ \exp[\sum_{i=1}^n \int dy \Omega(x_i - y) J_0(y)] \prod_{i=1}^n \tilde{\psi}(x_i)|0\rangle \right\} \tag{28a}$$

where,

$$\tilde{\psi}(x) = e^{-i \int_\infty^x dy_i A_i(y)} \psi(x) \tag{28b}$$

Now the general structure of the state functional of a system of n-particles with statistics σ following from the representation theory of the Braid group is given by,

$$\psi_\sigma[\chi(x_1)\cdots\chi(x_n); t] = \exp[2i\sigma \sum_{j=1}^n \sum_{i=1}^{j-1} \omega(x_i - x_j)] \psi_0[\chi(x_1)\cdots\chi(x_n); t] \tag{29}$$

where $\omega(x-y)$ is the multivalued polar angle of the vector $x-y$,

$$\omega(x-y) = \arctan \frac{x^2 - y^2}{x^1 - y^1} \tag{30}$$

and ψ_0 represents an n-particle functional with Bose statitistics. Since the expression in the curly brackets of eq.(28) represents a gauge invariant functional with commuting cocycles (because J_0 commutes with itself), it may be associated with ψ_0 of eq.(29). A complete equivalence between the two equations (28) and (29) can be established if we identify,

$$\Omega(x-y) = -2i\sigma\omega(x-y) \tag{31}$$

It is possible to give an explicit form for σ by making a connection of our approach to the Lagrangian (path integral) formalism of ref.(3). First observe that the action (1) may be expressed as,

$$S = S_{\text{matter}} + \int d^3x [J_\mu A^\mu + \frac{\theta}{4\pi^2} \epsilon^{\mu\nu\lambda} A_\mu \partial\nu A_\lambda] \tag{32}$$

Path integrating over A_μ leads to the Hopf term,

$$S = S_{\text{matter}} + \frac{1}{4\theta} \int d^3x d^3y J_\mu(x) \epsilon^{\mu\lambda\nu} \frac{(x-y)_\lambda}{|x-y|^3} J_\nu(y) \tag{33}$$

whose effect, as can be shown by mimicing the analysis of ref.(3) *, is to induce an arbitrary spin s,

$$s = -\sigma + \frac{1}{2} \quad , \quad \sigma = -\frac{1}{\theta} \tag{34}$$

on the physical states given by (29). Combining (31) and (34), we obtain from (21) the final expression for the one particle creation operator,

$$\hat{\psi}(x) = e^{2i(s-\frac{1}{2})\int dy\,\omega(x-y)J_0(y) - i\int^x dy_i A_i(y)}\psi(x) \tag{35}$$

which is multivalued due to the occurence of $\omega(x-y)$.

We now associate $\hat{\psi}(x)$ with the anyon operator of the model since it creates physical states (29) with arbitrary spin $s = s(\theta) = \frac{1}{\theta} + \frac{1}{2}$.This is a new construction for the anyon operator. It is gauge invariant so that the observed effects are physical. Note that earlier papers[4,6,8,9] discussed anyonicity of gauge dependent objects using gauge fixed Hamiltonian methods so that their interpretation remains obscure. The statistics of the anyon field is analysed by computing the product $\hat{\psi}(x)\hat{\psi}(y)$ which yields, on using (27) and the basic anti-commutator among the fermion fields,

$$\begin{aligned} \hat{\psi}(x)\hat{\psi}(y) &= -e^{2i(s(\theta)-\frac{1}{2})(\omega(x-y)-\omega(y-x))}\hat{\psi}(y)\hat{\psi}(x) \\ &= e^{\pm 2is(\theta)\pi}\hat{\psi}(y)\hat{\psi}(x) \end{aligned} \tag{36}$$

since $\omega(x-y)-\omega(y-x) = \pm\pi$ and the sign ambiguity occurs because the function $\omega(x-y)$ is defined only mod 2π. It is essential to preserve the consistency of the above relation. Physically it represents the arbitrariness present in the exchange of two particles which may be done either by a clockwise or an anticlockwise rotation[4]. We find, therefore, that the anyon operator obeys graded commutation relations. For integral values of the spin factor $s(\theta)$ (corresponding to bosons) commutators are obtained. Similarly anticommutators are realised for half integral (fermionic) values of $s(\theta)$. Hence the usual spin statistics theorem valid for bosons and fermions is reproduced. We note that the earlier construction[4] of the anyon operator was incompatible with this theorem[3].

We now show that the introduction of the anyon operators (35) influences the dynamics of the theory in a non trivial way. Consider the interaction piece of the Hamiltonian (eq.7),

$$\mathcal{H}_I = -\overline{\psi}\gamma_k(\frac{i}{2}\overleftrightarrow{\partial}^k + A^k)\psi \tag{37}$$

and express it in terms of the hat variables (35),

$$\mathcal{H}_I = -\frac{i}{2}\overline{\chi}\gamma_k\overleftrightarrow{\partial}^k\chi \tag{38a}$$

where,

$$\chi(x) = e^{-2i(s-\frac{1}{2})\int dy\,\omega(x-y)\hat{J}_0(y)}\hat{\psi}(x) \tag{38b}$$

* Note that in this paper the algebra is for scalars so that the additive factor of $\frac{1}{2}$ in eq(34),coming from the spin of the fermions,does not appear there.

implying that the interaction has been eliminated. This illustrates the dual nature of the C.S. theory, expressed either in terms of single valued fields with normal spin- statistics having gauge interactions or in terms of multivalued (hat) fields with arbitrary spin-statistics but without the gauge interactions[4,10]. In arriving at this result we have avoided the formal manipulations with multivalued functions which were done earlier[4] and subsequently criticised[3-7].

To conclude, we have shown that the C.S.theory coupled to fermionic matter fields can be consistently quantised in the Hamiltonian formalism without gauge constraints. The ambiguities associated with gauge fixing[4,5,6,9] are, thereby, avoided. All the space time symmetries of the theory as well as the complete Poincare algebra are preserved. The determination of the Fock space leads to the construction of multivalued anyon operators which create the physical states with arbitrary spin. These operators also satisfy graded commutation relations compatible with the spin-statistics theorem valid for bosons and fermions. The structure of the anyon operator given here is new. It is gauge independent while the conventional ones found in the literature[3,4,6,8,9] are not. This is important since any viable anyon operator must be gauge independent so that the observed effects are physical and not mere artifacts of the gauge. The connection of our Hamiltonian formalism with the Lagrangian (path integral) formulation[3] using the Hopf term has been illuminated. Finally we show that the effect of the anyon operator is to eliminate the gauge interaction, thereby reproducing the dual interpretation of the C.S. theory[4,8,10] but without employing ambiguous manipulations [4,8] with multivalued functions. A similar analysis for C.S. theory coupled to complex scalars led to identical conclusions[10]. The effects of including a Maxwell term in the theory and the implications of gauge fixing will be considered elsewhere.

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