UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-983

ENERGY-MOMENTUM TENSOR: METRICAL AND CANONICAL

H. Fleming
Instituto de Física, Universidade de São Paulo

Maio/1992

Energy-momentum tensor : Metrical and Canonical

H. Fleming
Instituto de Física
University of São Paulo

Abstract

The relation between these two types of energy-momentum tensor is explained in a way that is easily appended to most text-book treatments.

1 INTRODUCTION

It is well known that the canonical energy-momentum tensor [1] of a classical field theory is not symmetric in its indices, except for zero-spin fields, and that this spoils the elegance of the formalism by requiring an ugly expression for the angular momentum density of the field. This is clearly exposed in many places, like, for instance, [2], and the solution is given in the classical works of Belinfante [5] and of Rosenfeld [6]. All the matter is scholarly settled there, though strictly in a research report style, not appropriate for inclusion in a set of lectures using standard texts, like Landau, Lifshitz [1] or Jackson [4].The rules for replacing the canonical energy-momentum tensor by a symmetrical, equivalent one, the so-called Belinfante-Rosenfeld tensor, however, are quite simple, and deservedly well-known. A possible alternative is to use the so-called metrical energy-momentum tensor, introduced by Hilbert in his classical paper [7]. In this note we intend to elucidate in a simple way when these two kinds of energy-momentum tensor are equivalent and when they are not. We will restrict our treatment to a scalar field ϕ. The extension to other cases is obvious. This paper purports to be a pedagogical one.

2 QUESTIONS OF EQUIVALENCE

In order to properly introduce the metrical energy-momentum tensor we must work in curvilinear coordinates. Let $\mathcal{L}(g^{ij}, \frac{\partial g_{ij}}{\partial x^l}, \phi, \partial_i\phi)$ be a Lagrangian density.

The action is given by

$$S = \int d^4x\sqrt{(-g)}\mathcal{L} \tag{1}$$

The metrical tensor is obtained [1] by exploiting the fact that S must be invariant under infinitesimal coordinate transformations $x^i \to x'^i$, with

$$x'^i = x^i + \xi^i(x). \tag{2}$$

Fields and the metric respond to this transformation in the following way [1]:

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) = -\xi^i(x)\partial_i\phi \tag{3}$$

$$\delta g^i k(x) = \xi^{i;k} + \xi^{k;i}. \tag{4}$$

This induces in the action S the variation

$$\delta S = \int d^4x\delta(\sqrt{(-g)}\mathcal{L}) + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L} \tag{5}$$

where the second integral is essential, as a general coordinate transformation doesn't have to vanish at the boundaries of the integration domain. For a nice derivation of this term see [3] . It is his equation (170). Actually, this surface term is the key to the proof, as will be shortly seen. More explicitly,

$$\begin{aligned}\delta S &= \int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_l\delta\phi\} \\ &+ \int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}}\delta g^{ij} + \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\frac{\partial g^{ij}}{\partial x^l})}\frac{\partial}{\partial x^l}\delta g^{ij}] \\ &+ \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{6}$$

The usual partial integrations lead to

$$\begin{aligned}\delta S &= \int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial\phi} - (1/\sqrt{(-g)})\partial_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}]\}\delta\phi + \\ &+ \int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}} - \partial_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}]\delta g^{ij} + \\ &+ \int d\sigma_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}]\delta\phi \\ &+ \int d\sigma_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}\delta g^{ij} + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{7}$$

For $\phi(x)$ satisfying the equations of motion the first integral vanishes. Defining [1] the metrical energy-momentum tensor T_{ij} by

$$\frac{1}{2}T_{ij}\sqrt{(-g)} \equiv \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}} - \partial_l\frac{\partial(\sqrt{(-g\mathcal{L}}))}{\partial(\partial_l g^{ij})} \tag{8}$$

one has

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}\delta g^{ij} + \int d\sigma_l\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\delta\phi + \\ &+ \int d\sigma_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\frac{\partial g^{ij}}{\partial x^l})}\delta g^{ij} + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}\end{aligned} \tag{9}$$

Assume for a moment that $\mathcal{L}$ does not depend on the derivatives of g^{ij}. This means that the connection coefficients $\Gamma^i_{jk}$are not present, either explicitly or inside curvature tensors.(Of course this is always the case in Minkowski spacetime described by ' 'cartesian" coordinates).Inserting into (9) the values of $\delta\phi$ and δg^{ij} one has

$$\delta S = \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) + \int d\sigma_l\sqrt{(-g)}\xi^m\{-\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_m\phi + \delta^l_m\mathcal{L}\}, \tag{10}$$

that is,

$$\delta S = \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) - \int d\sigma_l\sqrt{(-g)}\xi^m)\Theta^l_m \tag{11}$$

where we recognize

$$\Theta^l_m = \frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_m\phi - \delta^l_m\mathcal{L}$$

as the canonical energy-momentum tensor. Now, as shown in detail by [1],

$$\frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) = -\int d^4x\sqrt{(-g)}T^k_{i;k}\xi^i + \int d\sigma_l\sqrt{(-g)}T^l_m\xi^m \tag{12}$$

Taking (12) into (11),

$$\delta S = -\int d^4x\sqrt{(-g)}T^k_{i;k}\xi^i + \int d\sigma_l\sqrt{(-g)}\xi^m(T^l_m - \Theta^l_m). \tag{13}$$

As δS should vanish for arbitrary ξ^i, one has

$$T^k_{i;k} = 0 \tag{14}$$

and

$$\int d\sigma_l\sqrt{(-g)}(T^l_m - \Theta^l_m) = 0$$

or

$$\int d\sigma_l\sqrt{(-g)}T^l_m = \int d\sigma_l\sqrt{(-g)}\Theta^l_m. \tag{15}$$

Finally, in the particular case of affine Minkowskian coordinates, one has

$$\partial_k T^k_i = 0$$

$$\int d^3x T^0_m = \int d^3x\Theta^0_m \tag{16}$$

showing the equivalence in the sense of Belinfante-Rosenfeld. Notice that the equivalence expressed in (16) is always true in Minkowski (described by affine coordinates). That expressed by (15), on the contrary, valid for the case of curved space times as well, is only true when there is no dependence in $\partial_l g^{ij}$. This is the basic criterion.

3 CONCLUSION

It is a common, and efficient, practice to get the energy-momentum tensor of a theory directly from the metrical one. Arguments to the effect that this is equivalent to taking the Belinfante-Rosenfeld tensor are usually omitted. As shown here , a slight modification of the formalism (I am thinking of Landau's), and a correct book-keeping of the surface terms, proves the equivalence as a bonus. To summarize: in Minkowski spacetime metrical and symmetrized-canonical tensors are equivalent. In curved spacetimes, not always. Eq.(9) provides the basis for the analysis of the equivalence in all cases.

It is my pleasure to thank Professor J. Frenkel for suggestions , criticism and encouragement.

References

[1] L.D.Landau, E.M. Lifshitz, *The Classical Theory of Fields* 4th. Edition, Pergamon Press.

[2] F. Rohrlich, *Classical Charged Particles* , Addison-Wesley, Reading, 1965.

[3] W. Pauli, *Theory of Relativity*, Pergamon Press, London, 1958.

[4] J.D. Jackson,*Classical Electrodynamics*, Wiley.

[5] F.J. Belinfante, *Physica*,**6**(1939) 887.

[6] L. Rosenfeld, *Mém.Acad.Roy.Belg.(Cl.Sc.)*18(1940)6.

[7] D. Hilbert, *Grundlagen der Physik*, in **Gesammelte Werke** , Springer, Berlin.