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DYNAMICAL SYSTEMS AND DETERMINISTIC CHAOS IN PLASMA PHYSICS

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Abstract — After a short introduction into the theory of dynamical systems and chaos, recent applications to Plasma Physics are reviewed, including both Hamiltonian and dissipative typical systems describing plasma phenomena.

Introduzem-se sucintamente elementos da teoria de sistemas dinâmicos e caos determinístico e resenham-se aplicações recentes à Física de Plasmas. Consideram-se tanto sistemas Hamiltonianos como sistemas dissipativos descrevendo fenômenos típicos em plasmas.

1. Introduction

Plasma Physics shows a very rich variety of non-linear phenomena and instabilities (1) which are already present in several important applications including Thermonuclear Controlled Fusion, Solar Physics, and Astrophysics. Because of this richness, Plasma Physics is an appropriate laboratory to discuss some of the concepts from the theory of dynamical systems and chaos. This discussion is the main goal of this review paper.

A growing interest in non-linear studies has been observed in the last decades. New methods and concepts, both in *conservative (Hamiltonian)* (2,3) and *dissipative* (4,5) systems, have been introduced. The central feature of the new developments is that deterministic chaos can be already found in systems with at least three degrees of freedom. This fact has significantly changed the previous picture of turbulent processes, particularly in Physics of Fluids.

Non-linear Hamiltonian systems behave as integrable systems and, under certain conditions, show chaotic or stochastic behavior. Examples of this last dynamics in Plasma Physics are the motion of charged particles in the presence of electric or magnetic fields and the stochastic behaviour and diffusion of magnetic field lines.

Most of the differences concerning previous approaches are found within the domain of dissipative systems. For these systems, a mathematical theory of chaotic processes was established, in particular introducing the concept of *chaotic attractor* (4,5) and the associated notion of *sensitive dependence on initial conditions.* The study of topological properties of chaotic attractors reconstructed from experimental temporal series has giving rise to an alternative interpretation of turbulent processes, where *coherent dissipative structures* would play a central role. The study of the order-chaos transition in dissipative systems has led to the identification of sequences of bifurcations, including certain universal aspects, which have been observed in several experiments scattered through different disciplines. Examples of dissipative chaotic dynamics in Plasma Physics are provided by experimental studies of routes to chaos, the non-linear coupling of waves, and the low dimensionalities associated with broadband chaotic experimental signals.

The complexity of the subject and the variety of applications do not permit an exhaustive review of the topic-title. I restrict myself to a general view of the subject considering specific topics. In Section 2, the main ideas related to the above referred developments are shortly presented. In Section 3, some recent applications to plasmas in Hamiltonian dynamical systems are discussed, including the stochastic behavior and diffusion of toroidal magnetic field lines, and the chaotic guiding-centre diffusion of charged particles moving around a single chain of magnetic islands in the presence of a single prescribed low-frequency electrostatic wave. Section 4 reviews the domain of dissipative dynamical systems discussing results for routes to chaos in laboratory plasmas, correlation dimension for electron density fluctuations measured in Tokamaks, non-linear coupling of waves, and certain analogies between magneto-hydrodynamic interchange instabilities and the Rayleigh-Bénard convection. Conclusions are presented in Section 5.

The present review does not exhaust the subject. Its descriptive nature, opposed to rigour, is intentional. The text below and the references *in fine* intend to give to the non--specialist reader an idea of the developments as well as suggestions for further studies.

2. Fundamentals

Electromagnetism, Quantum Mechanics and Relativity have dominated Physics during most of this century. The range of applicability of these disciplines was extended, with appropriate techniques in perturbation theory, to include non-linear phenomena. In spite of the success, linearization does not answer all the questions. Several important problems, *e.g.*, the asymptotic stability of dynamical systems, were left without solution.

The tradition of dynamical system studies comes back to Henri-Poincaré (1854–1912). Inspired by problems in Celestial Mechanics (6), Poincaré noticed the utility of the study

of topological structures in the phase space of dynamical trajectories. G.D. Birkhoff (1884–1944) (7) gave important contributions to the Ergodic Theory and the foundations of Statistical Mechanics, reinforcing the theoretical basis of Poincaré. However, only during the last thirty years the "Science of the Non-Linear" has been established with a certain autonomy. A significant role for this transformation was played by different disciplines, as the Theory of Critical Phenomena, Quantum Optics and Fluid Dynamics.

The theory of dynamical systems has been developed along different directions. One direction, related to *quasi*-integrable systems, is associated with the so called *KAM theorem* (8,9) (stated by Kolmogorov, demonstrated for fluxes by Arnold, and for maps by Moser). This theorem says that multiple periodic systems, obtained by a sufficiently small perturbation of an integrable system (for this the orbits are always over smooth surfaces, *i.e.*, invariant *tori* determined by the constants of motion), have trajectories over the invariant *tori* of the respective integrable system, once the initial conditions are sufficiently far away from the resonances of the system. These invariant *tori* are destructed by sufficiently large perturbations. Following Chirikov (10), the practical method to estimate the perturbation amplitude necessary for the destruction of the invariant surfaces is the overlapping of resonances. When this overlap occurs, regions of stochasticity (resonant layers) near the separatrix are formed, and global stochasticity appears when primary resonant layers meet each other.

In order to study dissipative dynamical systems, fluxes and maps with contracting phase space volumes, new concepts have been introduced: the *strange (chaotic) attractor* (11,12), and its corollary, the *sensitive dependence on initial conditions* (d.i.c.). Lorenz, a meteorologist interested in the problem of weather forecasting for long times, was the first researcher to observe that intrinsically chaotic motions can be found in dissipative deterministic systems (13). Lorenz considered the equations associated with two-dimensional thermal convection to conclude for the impracticability of such a prevision, because of imprecisions in the determination of initial conditions.

Chaotic behavior can already be observed in systems with at least three degrees of freedom. This fact changes radically the picture of the order-turbulence transition in fluids: turbulent behavior can also be reached in systems with dynamics completely represented in a low dimension phase space. The traditional picture of this transition considers turbulence as a hierarchy of instabilities. According to the scenario proposed by Landau (14), a succession of unstable modes, with incommensurable frequencies, appears as the *control parameter* (typically a parameter proportional to the driving force of the system, *e.g.*, voltage, temperature, electric or magnetic field; in Fluid Dynamics: Reynolds number, Prandtl number, etc.) is increased. As a consequence, the system will present more and more complicated patterns, but not strictly chaotic, since the correlation time is finite for a finite number of waves. However, Landau's model does not agree with experiments. It does not give the continuous spectra which is observed in fluid turbulence, one of the features of chaotic dynamics, and does not exhibit d.i.c., another feature of deterministic chaos.

The study of routes to chaos has its roots in the theory of differential equations (*theory of bifurcations*) and constitutes the so called *geometric theory of chaos* (15).

As a result of the variation of the control parameter in a dynamical system, the associated asymptotic motion may change. The values of the control parameter where there is a change of asymptotic regime are called *bifurcation points*. Sequences of bifurcations may be seen as *scenarios*. The problem of the possible scenarios or *routes to chaos* is basically to understand how a periodic regime can loose stability. Depending on the way this occurs, and the type of bifurcation involved, the dynamical behavior that will substitute the periodic regime (now unstable) will be different.

The three main scenarios for the transition order-chaos are the *Ruelle-Takens scenario*, *via quasi*-periodicity, the *Feigenbaum scenario*, via period doubling, and the *Pomeau--Manneville scenario*, *via* intermittency.

The Ruelle-Takens scenario (11,16) is a route to chaos based on bifurcations of *tori*. It includes: (a) initially the system is in a stationary state (*e.g.*, laminar flow in a fluid); (b) for

a certain value of the control parameter, the system experiences a Hopf bifurcation (15) and starts to oscillate with frequency f_1; (c) by further increasing of control parameter the system experiences a Neimark bifurcation (15) and oscillates with two independent frequencies, f_1 and f_2 ; (d) by an additional increase of the control parameter, there is a second Neimark bifurcation producing an additional frequency f_3. Then, the *torus* T^3 resulting from this last bifurcation may, under general conditions, become unstable, with the appearance of a strange (chaotic) attractor and a broadband continuous spectrum.

The Feigenbaum scenario (17,18) to chaos is the best known and studied scenario, being supported by several experimental evidences. The archetype for the study of this scenario is the unimodal maps with negative Schwarzian derivative (4). These maps show similar patterns of bifurcation, and chaos is reached through an infinite cascade of flip (period doubling) bifurcations (15). The values of the control parameter μ for successive bifurcations form a rapidly increasing convergent series towards an accumulation point μ_∞ (not universal) which can only be obtained numerically. Beyond this point chaos appears. Embedded in the chaos there are windows of periodicity (also odd periodicity) and inverse cascades (39).

Dynamical systems with Feigenbaum scenario display certain universalities: scaling laws for the bifurcation points μ_n (n = 1, 2, ...) and for the distances d_n between the fixed points nearest to the critical point (maximum of the unimodal map) (15). The following scaling laws are verifyed, $\mu_n = \mu_\infty - \text{cte}\ \delta^{-n}$, $d_n/d_{n+1} = -\alpha\ (n \gg 1)$, with the universal values $\delta = 4.669201609...$ and $\alpha = 2.502907875...$ for systems showing the Feigenbaum route to chaos. There is an analogy between these scaling laws and the theory of phase transitions. Using the renormalization group terminology, the constants δ and α can be considered as "*critical exponents*" at the accumulation point μ_∞.

Pommeau and Manneville (20) proposed three mechanisms for the appearance of chaos, which are related with intermittency. Intermittencies are characterized by a signal which is regular, periodic or *quasi*-periodic, during a certain time interval, and evolves to produce, in a short period, a generally chaotic "burst". This burst is followed by a phase of regular oscillations interrupted again by another burst of a periodic oscillations. This pattern repeats continuously. The global chaotic behavior is given by the burst, but mainly by the random distribution of the lengths of the regular periods.

Other scenarios to chaos exist (21). They have in common the fact that chaos is initiated by bifurcations. The different scenarios do not contradict each other; instead, they develop themselves concurrently in different regions of the phase space (22).

The characterization of chaos found in experiments is an important problem in the theory of chaos. The level of complexity of a dissipative system can be classified studying the geometric structure of the associated attractor. Dissipative systems showing chaotic dynamics, except rare cases, show a *strange attractor* (chaotic, *i.e.*, with d.i.c.) with *fractal* (*i.e.*, non integer) dimension (23). Classical methods (Fourier transform and auto-correlation functions) do not permit the distinction between deterministic chaos and white noise. The use of *Poincaré sections* (4,15) or phase portraits is useful only for dynamics completely represented in a phase space with dimension less than or equal to 3.

The theory of dynamical systems has provided new tools to analyze experimental chaotic temporal series. When dealing with experiments, it is generally not possible to have access to the m simultaneous signals necessary to describe the trajectory in the m-dimensional "real" phase space. In fact, the temporal dependence of only one scalar variable $x(t)$ is often monitored. A theorem, due to Takens (24,25), permits the reconstruction of the dynamics in a pseudo-phase space, in general topologically equivalent to the trajectory in the real phase space.

Two general approaches have been applied to the description and characterization of strange attractors. The *metrical* approach is related to dynamical information and the *topological* approach is based on the properties of periodic orbits which are embedded in the strange attractor. Metric properties provide information about expansion rates of initially near trajectories (*Lyapunov exponents* (11,26)) and about the rate of production of information in the system (*Kolmogorov-Sinai entropy* (4,27)). One positive Lyapunov exponent means d.i.c. The Kolmogorov-Sinai entropy is a measure of the global degree of chaos in the system. Deterministic chaos is characterized by a non-zero finite Kolmogorov-Sinai entropy. The metrical characterization also includes the statical-statistical approach, where global informations concerning the local structure of attractors are obtained. In most of the cases, chaotic attractors are characterized by a *fractal* (23) or *multifractal* measure (28), which can be analyzed using the *generalized dimensions* (28,29) or the *spectrum of singularities* (28,30). The metrical characterization is invariant under coordinate changes but depends on control parameters. The topological approach is independent of coordinate changes and remains invariant under variation of the control parameter. It provides information about the organization of the strange attractor and uses the *symbolic dynamics* (5,31) as a basic tool.

There is a great variety of algorithms (32) to analyze experimental signals. These algorithms do not constitute a precise body. The application to experimental data involves certain aspects not completely elucidated and, apparently, optimistic error estimates.

These algorithms include procedures to calculate the spectrum of Lyapunov exponents (33), the Kolmogorov-Sinai entropy (29), the correlation dimension (generalized dimension of order two) (29), and the spectrum of singularities (30,34). Noise reduction and reconstruction of the dynamics from temporal experimental series are topics related to the problem of chaos characterization. However, in spite of some proposals in the literature (35,36), these problems are not yet completely solved.

This short résumé of fundamentals in dynamical systems and deterministic chaos theory is very incomplete and fragmented. For those interested in additional details I suggest the following books, classified according to the degree of complexity in elementary (15,19,37,38), intermediary (2,5,27,39–43), and advanced (3,46,47).

3. Plasmas as conservative systems

The structure and geometry of magnetic fields in confined plasmas are determining elements of the behavior of these systems. The transport of charged particles along magnetic field lines is much faster than across them. The destruction of the magnetic surfaces and the formation of regions of stochastic field lines, as a consequence of external currents as well as auto-consistent currents, produces the *degradation of confinement.*

These studies have experienced remarkable development through the use of techniques from the theory of low dimension dynamical systems. Historically, Kadanoff (48) showed that Feigenbaum's ideas concerning renormalization, in the context of period doubling, could be applied to the problem of the existence of confining magnetic surfaces. This programme was subsequently carried out by Mackay (45).

In systems with *axisymmetric toroidal geometry,* like Tokamaks, the Hamiltonian describing the magnetic field lines is *integrable* (50,51). When the toroidal component exists, it is convenient to introduce, as a function of the position $\vec{r}$, the following natural magnetic coordinates: the toroidal flux function $\phi(\vec{r})$; the toroidal angle $\xi(\vec{r})$; and the poloidal angle $\theta(\vec{r})$.

The magnetic field can be written, in the contravariant form, as (51)

$$\vec{B}(\vec{r}) = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\xi \times \vec{\nabla}\psi_p \quad ,$$

where $\psi_p = \psi_p(\vec{r})$ is the poloidal flux function. The field line equations are given by

$$\frac{d\theta}{d\xi} = \frac{\partial\psi_p}{\partial\psi} \quad , \quad \frac{d\psi}{d\xi} = -\frac{\partial\psi_p}{\partial\theta} \quad . \tag{1}$$

They are *canonical equations of motion* for the Hamiltonian ψ_p , where ξ is the temporal variable. For axisymmetry, ψ is independent of ξ , and the magnetic field lies on the so-called *magnetic surfaces* given by $\psi =$ cte, which are *tori.*

The field lines can be studied using *Poincaré sections* (2,4); the problem of following field lines can be so reduced to a map of the surface of section on it (*Poincaré map*). For integrable

systems the map so generated is conservative, which is equivalent to area conservation of the Poincaré invariants for Hamiltonian systems. Orbits of field lines for perturbed non--axisymmetric systems can also be reduced to maps using an analogous procedure. If the perturbation is not very large, the Poincaré map is similar to the non-perturbed map.

As a consequence, it is possible to identify field lines with orbits of a dynamical system in a three-dimensional space. Reasoning in this way, the behavior of field lines is so generic as the motion of particles in a system represented by an Hamiltonian with $1\frac{1}{2}$ degrees of freedom. Also, another problem, the *guiding-centre motion* of particles in a Tokamak, can be studied using methods from the theory of dynamical systems.

Chaotic behavior can be found in magnetic field lines or in guiding-centre motion. This stochastic behavior can be: partial, for small perturbations, when some magnetic surfaces around the rational surfaces are eventually destroyed and substituted by regions of stochastic field lines; or, global, when these regions are sufficiently large to produce overlap (*Chirikov criterion* (10)), producing large scale stochasticity. Additional details, in an introductory level, can be found in Lichtenberg and Lieberman (2,52). The connections among renormalization, period doubling and destruction of magnetic surfaces are discussed by Greene (53). The use of non-canonical phase space coordinates, in the context of the description of magnetic field lines fluxes, is considered by Littlejohn (54). The diffusion of magnetic field lines in toroidal geometry and the chaotic diffusion of guiding-centres are discussed, as examples, in the following Section.

3.1. Stochasticity and diffusion of magnetic field lines

The fact that magnetic field lines can become stochastic as a result of the overlap of two or more chains of magnetic islands is a result known for more than two decades (55). In Tokamaks, this occurs as a result of two helicoidal resonant perturbations or only one perturbation with toroidal effects.

In the last years, several applications of these ideas have been done, in particular to the problem of confinement degradation with the increase of power in Tokamaks. Rebut *et al.* (56) propose a theoretical model for this degradation. The topology of the magnetic field used by the authors is represented by a chain of magnetic islands, resulting from the representation of the magnetic field in a slab geometry with torsion. By relating the flux of heat with the magnetic perturbation level (large poloidal mode numbers), they conclude that significant power can be transported through the plasma by low level magnetic fluctuations. The authors conjecture also, as a possible mechanism for the self-maintenance of the magnetic islands, the difference of resistivity between chaotic zones and magnetic islands, as a result of the magnetic field line trajectory and the presence of fast electrons in the chaotic zones.

Caldas *et al.* (57,58) consider the effect of perturbative resonant helicoidal currents (produced by external helicoidal windings) in the topology of the magnetic fields in Tokamak plasmas. These effects can be used both to produce helicoidal perturbations to inhibit undesirable plasma oscillations or to produce effects of magnetic ergodic limiter. In the first case, for helicoidal currents much smaller than the plasma current, it is observed partial stochasticity in the Poincaré section, with the appearance of chaotic regions between the magnetic islands. For the ergodic magnetic limiter, the integration of the perturbed magnetic field lines gives maps that show intermittent transition between chaotic and *quasi*-periodic regimes.

The radial diffusion of magnetic field lines in toroidal geometry is considered by Mendonça (59) in a recent paper. The originality of the author is to extend a previous formalism

(*renormalized theory of turbulence*), used by Krommes *et al.* (60) and Misguich *et al.* (61), to study the guiding-centre diffusion of charged particles in magneto-plasmas in presence of broadband electrostatic turbulence. In Mendonça's formalism, the guiding-centre equations are essentially substituted by the magnetic field line equations (eq.(1)).

The renormalized theory of turbulence uses Gaussian-like approximations, including cumulant expansion (up to second order) and the Corsin hypothesis (61). These approximations are employed by Mendonça (59), together with the hypothesis of stationary and poloidal homogeneity of the turbulence, to derive a general expression for the diffusion coefficient of the stochastic field lines. This expression is valid both in the limits of small and large torsion of the magnetic field. For small torsion the result agrees with the *quasi*-linear approximate solution. The author also discusses the conditions to reduce the exact differential equations for the field lines into discrete maps; the diffusion coefficient is written as a discrete map that reduces to the standard map (2) in an appropriate limit.

All these results in Mendonça's paper refer to situations where the magnetic field lines are already stochastic and the diffusion process occurs in extensive regions of the toroidal configuration. It is a stringent limitation concerning applicability of the results. In effect, for low level magnetic field fluctuations, the stochastic field lines are confined in thin layers between magnetic islands and the previously calculated diffusion coefficient has no sense. In order to overcome this limitation, the author proposes an alternative definition for the diffusion coefficient, valid also in the transition regions between local and global stochasticity. The pertinence and efficiency of this definition should still be verified.

3.2. Chaotic diffusion of guiding-centres

The motion of charged particles can become chaotic in the presence of a few electrostatic waves. Zalavsky and Filonenko (62), and Escande (63) have shown that two electrostatic waves are sufficient for this purpose. In this case the system is described by a $1\frac{1}{2}$ degrees of freedom non-autonomous Hamiltonian.

For the guiding-centre motion, two electrostatic waves propagating in the plane perpendicular to a strong magnetic field are not sufficient to produce chaotic motion (64); three plane waves are seen to be necessary (65).

Weyssow *et al.* (64) consider the problem of guiding-centre diffusion in the presence of an electric and a magnetic field. The fields are given by a model which assumes a chain of magnetic islands and only one low-frequency electrostatic wave propagating perpendicular to the non-perturbed magnetic field. For this last field a *slab model* with torsion is used as a local approximation for the *standard model* (66) for the toroidal magnetic field. The magnetic islands are introduced in the slab model with torsion as radial perturbations of the magnetic field. The electrostatic wave is chosen in order to be localized near the same rational surface as the magnetic island. It can be thought as a *drift wave* propagating in the plane perpendicular to the magnetic field with an angular frequency and wave number near the maximum of the spectrum. Experimental parameters from TFR and JET Tokamaks are used both for the magnetic island and for the electrostatic wave. The authors include in the guiding-centre equations the $\vec{E} \times \vec{B}$ drift, the parallel motion, and the curvature and magnetic gradient drifts. For low frequencies compared to the cyclotron frequency the polarization drift is not taken into account. Also, as a consequence of the assumed ordering, the curvature and magnetic drifts are seen to be of higher order. After these approximations it results a $1\frac{1}{2}$ degrees of freedom Hamiltonian, which is the paradigm Hamiltonian found in previous works (62, 63, 67). However, as the authors recall, their situation is different. Zaslavsky and Filonenko (62) consider a charged particle in the field of two electrostatic waves, and Rochester and Stix (67), magnetic field lines with helicoidal magnetic perturbations.

The main result of the paper by Weyssow *et al.* is the numerical verification that the guiding-centre motion in a chain of magnetic islands (slab geometry) can become chaotic in the presence of a single low frequency electrostatic wave. It is also verified that the chaotic diffusion is selective in the space of velocities; it is important mainly for ion velocities greater

than the thermal velocity, and for practically all the electron population, except those with very low parallel velocities (relativistic corrections were not taken into account). The authors conjecture a mechanism for diffusion contributing to the electron and suprathermal ion radial losses. The idea is that the diffusing process can be repeated in the radial direction through successive chains of magnetic islands in the radial direction; the resulting radial diffusion would contribute to the radial electric field and plasma rotation.

4. Plasma as dissipative systems

4.1. Routes to chaos

Scenarios for the transition order-chaos have been observed in several experiments in Plasma Physics.

Boswell (68) considers natural oscillations in an electron beam propagating parallel to a magnetic field in a low pressure gas. The beam current I is the control parameter. Chaos is observed after a sequence of period doublings in the amplitude of oscillations. A period--three window, with the period-six bifurcation, is observed increasing I. The author mimics the oscillations with a quadratic map, $F(x_n) = C - x_n^2$, that reproduces the main features of the observed pattern.

Bora *et al.* (68) report period-doublings in non-linear ion-cyclotron and lower-hybrid waves driven by a radio-frequency field near the lower hybrid frequency in the toroidal experimental device BETA. Using the antenna power as control parameter, a few period--doublings are observed, giving $\delta = 4.138$ for the Feigenbaum constant ($\delta_{Feig.} = 4.66$).

Braun *et al.* (70) study dc-excited discharges in low pressure discharges. The control parameter is the potential across the tube. The current through and the light from the discharge show period-doublings, chaotic behavior and windows of periodicity, corresponding to the typical dynamics found in unidimensional unimodal maps, like the logistic map $F(x_n) = \mu x_n(1 - x_n)$. It is also observed a scenario typical of two-dimensional maps, like the Henon map (4), where a period-doubling cascade is interrupted by a bifurcation with period multiplied by integers other than 2 (mainly 3 or 5). The authors conjecture the possibility that one-dimensional maps would not be sufficient to model such discharges.

Cheung and Wong (71) describe chaotic behavior in a pulsed plasma discharge in a large non-magnetized device. Monitoring the total current collected in the discharge camera, two scenarios to chaos are observed. The first, through the route $T \to IC \to 3T \to 2T$, where T is the period of the pulsed discharge $(= 2\pi/\omega)$, IC means an intermittent chaotic state, and $3T$ and $2T$ mean, respectively, periods 3 and 2. The second scenario is the Feigenbaum route: three successive bifurcations $(T \to 2T \to 4T)$ give $\delta = 4.4 \pm 0.3$ and $\alpha = 2.3 \pm 0.2$, which are in agreement with the values found by Feigenbaum.

In another paper, Cheung *et al.* (72) use a steady-state unmagnetized plasma device, consisting of an electron-emitting cathode at one end and a current-collecting anode at the other end, to observe intermittency in the anode current (the voltage is the control parameter). This intermittency seems to be of type-I (most probable laminar lengths are the longer), according to the terminology introduced by Pommeau and Manneville (20). Such indication follows from the experimental verification of low frequency noise ($f^{-2.1\pm0.2}$ for $f \gg 1$ kHz) at the threshold of the chaotic burst. Such a scaling was also observed in numerical simulations showing type-I intermittence, where Lorentzian spectra with scaling law f^{-2} were also observed.

Jing *et al.* (74) use a multipolar magnetic device to study a steady-state plasma produced by a *dc* discharge in Argon. The control parameters, which may be varied one at a time, are the gas pressure, the filament current and the discharge voltage. They monitored the electron density N_e, the plasma potential V_p, and the discharge current I_d. By varying the voltage, the authors report sequences of period doublings with the appearance of limit cycles in the $I_d \times N_e$ and $N_e \times \dot{N}_e$ phase spaces. Similar results are obtained by Qin *et al.* (75) in a plasma of the same type. In addition to period doubling in the principal sequence and evidences of intermittent behavior, the authors find period doublings in odd

period windows; the period-three window is seen to be the larger, showing resemblance with the bifurcation diagram of unimodal one-dimensional maps.

In spite of the interesting results reported above, showing a very rich dynamics, additional efforts should be done in order to model the experimental data. In particular, the reported dynamics that resemble two-dimensional maps (70,71) give important informations that should be taken into account. Recent techniques (35,36), used in the context of dynamics reconstruction from temporal experimental series, can be envisaged as attractive tools to attack this problem.

4.2. Correlation dimension

The statistical mechanics of turbulence is a very complex topic. In spite of the importance of turbulence in general, in particular in electromagnetic media, usually associated with practical problems like ***anomalous diffusion*** in magnetic confinement devices and astrophysical plasmas, there is no satisfactory theory for ***fully developed turbulence***. The ***quasi-linear*** approximations, in spite of giving the threshold for several instabilities, fail in several experimental situations of interest.

The theory of deterministic chaos gives alternative tools to analyze turbulent processes. In particular, given an experimental temporal series, and admitting the existence of an associated attractor, the ***Haussdorf dimension*** D_0 (fractal dimension) gives an estimate of the complexity of the turbulence, corresponding, in principle, to the smaller number of degrees of freedom necessary to describe the dynamics.

However, the available algorithms to determine D_0 are problematic for $D_0 > 2$. The Haussdorf dimension belongs to an infinite series of *generalized dimensions* (28,29) D_q, $q \in \mathbb{R}$. The knowledge of the complete series corresponds to the full characterization of the attractor. It was shown (76) that the dimensions D_q are ordered according to $D_q \leq D_{q'}$, $q' > q$, where the equality is valid for homogeneous attractors. It follows that D_0 is the *upper bound* for the D_q.

The *correlation dimension* $D_2 = \nu$ is available through the utilization of an algorithm due to Grassberger and Procaccia (G&P) (25,29). In general ν is very near to D_0.

The G&P algorithm, when applied to experimental data, includes initially the *Takens reconstruction* of the attractor (24). One constructs a set of *d*-dimensional vectors $\{\vec{\xi}_i\}$, with $\vec{\xi}_i = \{x(i), x(i+\tau), \ldots, x(i+(d-1)\tau\}$, where the $x(i)$ represent the experimental time data and τ is a time step of order of the signal auto-correlation time. The *embedding dimension* d is chosen sufficiently high in such a way that the phase space merges completely the attractor.

The distribution of points in the reconstructed trajectory (topologicaly equivalent to the "real" attractor) can be characterized by a *correlation integral* $C^d(l)$, which is proportional to the number of Euclidean distances between two points of the trajectory smaller than a correlation length l, *i.e.*, $C^d(l) = \frac{1}{N^2}\sum_i \sum_j \theta(l - |\vec{\xi}_i - \vec{\xi}_j|)$, where θ is the Heaviside function and N is the number of vectors $\vec{\xi}_i$.

In general, $C^d(l)$ shows, in the limit $l \to 0$, $N \to \infty$ and $d \to \infty$, a scaling law $C^d(l) \propto l^\nu$. Then, the slope of $\ln[C^d(l)] \times \ln(l)$, for sufficiently high N and d, converges to the dimension ν for increasingly embedding dimensions.

In practical applications N is finite. This imposes statistical limitations for the reconstruction at large embedding dimensions. In addition, experimental noise strongly reduces the efficiency of the G&P algorithm. Other limitations of the G&P algorithm are discussed elsewhere (25,32). In particular, to guarantee convergence to a dimension ν it is necessary that $d \geq 2\nu + 1$ (25,29). As a consequence, the G&P algorithm does not permit, in most of the practical situations, to compute dimensions larger than 7 or 8.

Before continuing, a word of caution should be given. The study of dissipative dynamical systems has permitted the construction of a theory for the *order-temporal chaos transition*. This theory allows the reconciliation of determinism and stochasticity in systems with a small number of degrees of freedom. However, the application of this theory when several modes of the system are in interaction is less evident. The *Ruelle-Takens theory* (11) emphasizes the

instabilities of trajectories in a strange attractor merged in an abstract *low dimensional* phase space. In Hydrodynamics or in electromagnetic turbulence, the expression "turbulence" has a precise meaning: it characterizes regimes that show random fluctuations both in time and in space. As suggested by Monin (77), the approach in terms of dynamical systems that assumes the projection of the dynamics in a small number of effective modes, perhaps does not take into account the loss of spatial coherence. The study of temporal chaos for systems with spatial order does not impose, in principle, any fundamental problem. This is not the case when temporal chaos and spatial disorder are simultaneously present. For these cases, procedures like the G&P algorithm should be regarded with caution.

There are several applications of the G&P algorithm to fluctuating signals in Tokamaks. However, these signals were obtained in different devices and using different diagnostics. Several physical variables were analyzed, *e.g.*, magnetic field fluctuations (78–82), sawtooth activity (79), and density fluctuations (79,83–86). In general it is not possible to draw a definitive or unique picture from these papers.

However, some of the results presented in the literature reinforce the idea that low dimensions can be found in fluctuating signals measured in Tokamaks. In particular, the correlation dimension of electron density fluctuations at the edge of the Tokamaks TBR-1 (86), TFR (84) and Tosca (79) show that $2.4 \lesssim \nu \lesssim 5.0$, depending on the analyzed signal. Magnetic fluctuations show that $2.0 \lesssim \nu \lesssim 7.0$. In some cases, no convergence was observed up to embedding dimensions between 8 and 10, which means that $\nu \gtrsim 10$.

A conjecture concerning the interpretation of the low dimensionalities found for the density fluctuations was recently advanced (86).

These low dimensionalities may be consistent with the existence of *coherent structures* (density blobs) together with waves at the edge of the plasma. This picture represents a more ordered configuration, where *self-organization* would play a relevant part in the dynamics of the process. It emerges from the *theory of dissipative density-gradient-driven turbulence* in Tokamak edges (87). According to this theory, the basic constituents of the steady turbulent state are *broadened collective resonances*, rather than waves or eigenmodes. The collective resonances are driven by emissions from localized density fluctuation elements (*blobs*) produced by gradient relaxation and destroyed by the relative $\vec{E} \times \vec{B}$ convection and parallel collisional diffusion. The density blobs resemble eddies in a turbulent fluid, rather than perturbations associated with linear waves. Hence, the stationary turbulent state is viewed as a "soup" compressing waves and eddy-like blobs, which can be thought of as *coherent micro-structures*.

Large scale coherent structures (blobs) with a long decaying phase were recently observed (88) in an implicit simulation of non-linear drift waves in a magnetic field with torsion. More recently, Iizuka *et al.* (89) use a linear Q-machine to study the non-linear interaction of flute oscillations with a convective cell. The interaction resembles a self-organization process: energy from the inherent waves, as well as from the cells, is transferred to modes with lower mode number; an inverse cascade in the spectrum occurs and an extended structure appears.

Some data for density fluctuations in the scrape-layer of Tokamaks continue, however, to show high dimensionalities. Additional theoretical and experimental efforts should be done in order to answer some questions: (a) are these high dimensionalities intrinsic to the signals, indicating a usual turbulent behavior?; or, (b) are they spurious in the sense that they were introduced by external factors like inappropriate frequency of sampling, electronic noise or filtering processes?

Assuming that external factors are unimportant and that the G&P algorithm gives trustable results, even for spatio-temporal chaos, a question remains: why some signals are "more complicated" (higher dimensionalities) than others? A possible interpretation for this result, in the case of edge plasma, is that the plasma experiences a sort of competition between the tendency toward self-organization and more complicated turbulent dynamics. Different signals, associated with the same physical variable, may have different histories. Some of them may evolve to more coherent structures, while others may not. Also, we are not able to control when the presumably more coherent structures appear. Some signals may

have been analyzed at a stage when they were "less turbulent". In other cases the analysis has been performed at a more turbulent stage. This complex dynamics results either in low dimensionality, with a picture related to a small number of degrees of freedom, or in high dimensionality, manifested in non saturation at low embedding dimensions. There remain, however, some doubts about the correctness of the application of the G&P algorithm to processes showing spatio-temporal chaos.

4.3. Non-linear coupling of waves

Relatively simple dissipative systems may show a rich mathematical structure. This richness is found in a great variety of experimental and theoretical situations.

An interesting example is given by the non-linear coupling of a linearly stable high--frequency wave to a smaller frequency damped wave, as considered by Meunier *et al.* (90) and reported by Maschke (91). Assume that both waves are characterized by complex amplitudes $A_{0,1} = |A_{0,1}| \exp(i\,\phi_{0,1})$, real frequencies $\omega_{0,1}$, growing rates $\gamma_0 > 0$ and $\gamma_1 < 0$, and are non-linearly coupled through the real coefficient V. Assume also that the waves satisfy the coupling equations

$$i\left(\frac{dA_0}{dt} - \gamma_0 A_0\right) = V A_1^2 e^{i\Delta\omega t} ,$$

$$i\left(\frac{dA_1}{dt} - \gamma_1 A_1\right) = V A_0 A_1^* e^{-i\Delta\omega t} ,$$

where $\Delta\omega = \omega_0 - 2\omega_1$. Changing variables (90), the following dissipative dynamical system is obtained:

$$\frac{dX}{dt} = X + \alpha Y - Z + 2Y^2$$

$$\frac{dY}{dt} = Y - \alpha X - 2XY$$

$$\frac{dZ}{dt} = -2\gamma Z + 2XZ \quad ,$$

where $\gamma = -\gamma_1/\gamma_0$, $\alpha = \Delta\omega/\gamma_0$ and $Z > 0$. Varying the parameters γ and α, the attractors experience normal and inverted Hopf bifurcations (4) and three regimes in the $\gamma - \delta$ plane are observed: stable equilibrium, non-linear saturation, and non-limited solution. In particular, intermittency is found in the transition from a limit cycle to a strange attractor. This behavior is also found in some experimental routes to chaos, as previously reported in Section 4.1.

In a recent paper, Nambu and Kawabe (92) propose a model for wave ionization with an external driving force. This model reduces to the well known Duffing equation (4,39) for the electron density modification. For typical values of the parameters the model shows chaotic behavior with type-I intermittency. This model, however, still waits for experimental verification.

Another application of concepts of dynamical systems to Plasma Physics has been developed by Maschke and Saramito (91,93). The authors study asymptotic solutions of visco-resistive-MHD equations (attractors) and their successive bifurcations as a parameter is changed. It is explored the formal equivalence between certain plasma instabilities, *e.g.*, interchange-unstable plasma layer in a curved magnetic field, and the problem of thermal instability of a fluid layer with a gradient of temperature (Rayleigh-Bénard convection (4,39)). The authors explore these analogies to identify routes to chaos through period doubling and bifurcation on *tori*.

5. Conclusions

The theory of dynamical systems is part of the so called "Science of Complexity", which includes procedures and methods, as well as an appropriate conceptual background, with

a large range of applicability: Physics, Chemistry, Ecology, Medicine, Economy etc. The identification of universal features and the study of mathematical analogies show the common aspects of different phenomena.

Plasma Physics, an essentially non-linear discipline, is appropriate to the exploration of the potentialities of these new developments from the theory of chaos and dynamical systems. Methods from this theory are powerful tools, *still in development*, not yet completely explored in Plasma Physics. Additional efforts should be done to test the analytical or computational models through experimental verification or construction of models to reproduce experimental results, *e.g.*, routes to chaos, low dimensionalities. In particular, it is important to test the relevance of these new ideas in the domain of fully developed turbulence.

The judicious application of the methods from the theory of chaos and dynamical systems should be seen as an exploratory and complementary tool to the more traditional approaches to non-linear plasma processes.

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