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F. E. M. Silveira,[1,a)] M. H. Benetti,[2,b)] and I. L. Caldas[3,c)]

AFFILIATIONS

[1]Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Rua Santa Adélia, 166, Bairro Bangu, CEP 09210-170, Santo André, SP, Brazil
[2]Instituto de Ciências Exatas, Universidade Federal Fluminense, Rua Desembargador Ellis Hermydio Figueira, 783, Aterrado, CEP 27213-145, Volta Redonda, RJ, Brazil
[3]Instituto de Física, Universidade de São Paulo, Rua do Matão, 1371, Cidade Universitária, CEP 05508-090, São Paulo, SP, Brazil

[a)]Author to whom correspondence should be addressed: francisco.silveira@ufabc.edu.br
[b)]mhbenetti@id.uff.br
[c)]ibere@if.usp.br

ABSTRACT

A detailed resource to data analysis shows that the widely known van Hoven and Derfler–Simonen laboratory results are far from reasonable agreement with the standard Bohm–Gross dispersion relation. We provide an extension of the usual notion of a polytropic index to non-Boltzmann–Gibbs statistics. Such an extension allows for the deduction of an equation of state of charged particles with the basis on the Kappa density distribution. That equation of state, in turn, enables suprathermal corrections to the standard dispersion relation. As a consequence, we prove that the employment of our suprathermal formula is in excellent agreement with the experimental data. Possible further applications of our theory are briefly addressed.

I. INTRODUCTION

The widely known Langmuir (or electron plasma) waves (or oscillations) were brought to light by Tonks and Langmuir in 1929.[1] Langmuir waves are usually fast oscillations of the electron concentration around its equilibrium value, n_0, in conducting media. Their linear frequency $\nu_{pe} \sim n_0^{1/2}$, where $[n_0] = \mathrm{m}^{-3}$ and $[\nu_{pe}] = \mathrm{Hz}$.[2] Therefore, the value $n_0 \sim 10^{18}\ \mathrm{m}^{-3}$, characteristic of electron equilibrium concentrations in several confined fusion plasma scenarios,[3–5] provides $\nu_{pe} \sim \mathrm{GHz}$. Actually, radiation propagating at ν_{pe} typically lies down in the microwave range of the electromagnetic spectrum. Furthermore, ν_{pe} depends weakly on the wavelength of the oscillation in cold plasmas.

In warm plasmas, the inclusion of a finite electron temperature, T, in the realm of Boltzmann–Gibbs statistics yields a dispersion of the frequency with the wavelength, which was theoretically predicted by Bohm and Gross in 1949.[6] Experimental verification of the Bohm–Gross dispersion was provided, in an independent way, by van Hoven[7] and Derfler and Simonen[8] in 1966. Although their results display all the basic features of electrostatic plasma waves, as we will show in Sec. IV below, data analysis reveals that they are far from reasonable agreement with the Bohm–Gross dispersion. Such outcomes strongly suggest the necessity for a non-Boltzmann–Gibbs approach to the propagation of Langmuir waves in warm plasmas. One possibility to fulfill this requirement is to look for corrections to the frequency-wavelength dispersion due to suprathermal effects.

Suprathermal effects were disclosed in 1968 by Vasyliunas,[9] who made use of the so-called Kappa probability distribution in order to describe phenomena occurring in space plasmas. The Kappa distribution may be regarded as a Maxwellian distribution, however, deformed by an elongated tail, which follows a power-law of a real and continuous parameter commonly referred to as the spectral κ-index.[10] Within that tail, high-energy particles are allowed, typically suprathermal electrons. This is why the Kappa distribution comprises an adequate theoretical ground for analyzing stationary regimes of low-density systems out of thermodynamic equilibrium, such as space plasmas, where binary collisions between particles are extremely rare events.[11]

The Kappa distribution has been employed in many studies of planetary magnetospheres[12–15] and solar winds.[16–20] Observational data collected by the Voyager spacecraft have suggested that certain ion properties exhibited in the outer heliosphere would be describable by the Kappa distribution.[21,22] The plasma properties of the Jupiter's magnetosphere have been continuously observed by the Jovian Auroral Distributions Experiment (JADE) on the Juno mission.[23] Based on a proposed connection of the Kappa distribution with non-extensive statistics,[24] a numerical analysis of JADE's data concerning ion density, flow, temperature, and composition has been recently performed,[25] and its accuracy-confidence correlation verified.[26]

In previous works,[27–29] we have provided an analytical formulation that allows for the introduction of a polytropic-like index, hence leading to the description of a generalized isothermal regime in the framework of non-Boltzmann–Gibbs statistics. In Ref. 27, an equation of state has been deduced in the realm of the Thomas–Fermi distribution, accounting for concentration discontinuities of alkali metals at high pressures. In Ref. 28, another equation of state has been derived, that time, with basis on the Kappa distribution, thus explaining a pressure profile observed in the Earth's magnetopause, in terms of solar wind particles. In Ref. 29, the possibility of emergence of systems with non-integer numbers of degrees of freedom has been explored through an analogy of the Kappa with Tsallis statistics.

In this work, we generalize the above referred formulation by extending the polytropic-like index notion to adiabatic processes occurring in non-Boltzmann–Gibbs statistics. As a by-product of that, we derive an equation of state of charged particles with a basis on the Kappa distribution, which enables suprathermal corrections to the usual Bohm–Gross dispersion. The corrected formula proves to be in excellent agreement with the van Hoven and Derfler–Simonen frequency-wavelength data. This paper is organized as follows.

In Sec. II, we start by briefly reviewing our aforementioned formulation of the polytropic-like index with a basis on the Kappa distribution, leading to a generalized isothermal regime. Then, such a formulation is extended to adiabatic processes. An equation of state of charged particles is derived. In Sec. III, that equation of state is employed to find out suprathermal corrections to the usual Bohm–Gross dispersion. It is shown that the suprathermal formula allows for reinterpretations of the number of degrees of freedom and electron Debye length in Kappa statistics.

In Sec. IV, resource to data analysis unveils that the van Hoven and Derfler–Simonen laboratory results are far from reasonable agreement with the usual Bohm–Gross dispersion. Thus, it is found out that the employment of the suprathermal formula proves to be in excellent agreement with the experimental data. In the concluding section, we summarize our work. Possible further applications of our theory are briefly addressed.

II. POLYTROPIC-LIKE INDICES AND KAPPA EQUATION OF STATE

The following discussion applies to any gas of charged particles. However, in order to simplify the understanding of our proposal, we regard an electron gas.

Consider a gas of electrons with mass m, charge $-e < 0$, and concentration n in the presence of a massive background of positive charge (essentially, a plasma). In the absence of a magnetic field, the time evolution of the gas flow $\vec{v}$ is determined through the gradients of the electrostatic potential Φ and isotropic pressure P developed in the medium by the equation of motion,

$$m\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = e\nabla\Phi - \frac{\nabla P}{n}. \tag{1}$$

On the assumption of an isothermal equation of state, the pressure gradient is described in terms of the concentration gradient by

$$\nabla P = k_{\rm B} T \nabla n, \tag{2}$$

where $k_{\rm B}$ is the Boltzmann constant and T is the (constant and uniform) Maxwellian temperature of the gas. It is widely known that the stationary state of the equilibrium of the system [the left-hand side of Eq. (1) vanishes] recovers the Boltzmann relation,[2]

$$n = n_0 \exp\left(\frac{e\Phi}{k_{\rm B}T}\right), \tag{3}$$

where n_0 is the electron concentration at a null potential amplitude (the magnitude of Φ falls-off very rapidly with the distance from the charges in a plasma).

Inspired by the aforementioned remark, in previous works,[27–29] we have provided an analytical formulation that allows for the introduction of a polytropic-like index, thereby leading to the description of a generalized isothermal regime in the framework of non-Boltzmann–Gibbs statistics. The starting point for that is the following. Let us replace Eq. (2) with

$$\nabla P = k_{\rm B}\Theta\nabla(\gamma n), \tag{4}$$

where Θ is a constant with the dimension of temperature (the precise meaning of Θ will be given in a moment) and γ is a function of the particle concentration n, $\gamma = \gamma(n)$, not the usual polytropic index. As a result of replacing Eq. (2) with Eq. (4), the stationary state of equilibrium of the system [again, the left-hand side of Eq. (1) vanishes] will be now determined by

$$\frac{\nabla(\gamma n)}{n} = \frac{e\nabla\Phi}{k_{\rm B}\Theta}. \tag{5}$$

In Ref. 28, we have chosen the particle concentration n to depend on the electrostatic potential Φ, $n = n(\Phi)$, through the Kappa density distribution:[9]

$$n = n_0\left[1 - \frac{(e\Phi)/(k_{\rm B}\Theta)}{(\kappa - 3/2)}\right]^{-(\kappa-1/2)}, \tag{6}$$

where, as found in diverse physical scenarios,[28–33] the Kappa Θ-temperature is given by

$$\Theta = \left(\frac{\kappa}{\kappa - 3/2}\right)T \tag{7}$$

in terms of the Maxwellian T-temperature, with the choice $3/2 < \kappa < \infty$ for the so-called spectral κ-index. In the limit $\kappa \to \infty$, Eq. (7) shows that $\Theta \to T$, and then Eq. (6) recovers Eq. (3).

By solving Eq. (6) for $\Phi = \Phi(n)$, we take its gradient to get the expression

$$\frac{e\nabla\Phi}{k_{\rm B}\Theta} = \left(\frac{\kappa - 3/2}{\kappa - 1/2}\right)\left(\frac{n}{n_0}\right)^{-[1/(\kappa-1/2)]-1}\nabla\left(\frac{n}{n_0}\right). \tag{8}$$

Subsequently, by substituting Eq. (8) in Eq. (5), we have the differential equation,

$$\nabla\left[\gamma\left(\frac{n}{n_0}\right)\right]=\left(\frac{\kappa-3/2}{\kappa-1/2}\right)\left(\frac{n}{n_0}\right)^{-1/(\kappa-1/2)}\nabla\left(\frac{n}{n_0}\right). \tag{9}$$

Finally, by integrating Eq. (9), under the boundary condition that the function $\gamma=\gamma(n)$ achieves its maximum value when the argument n attains its minimum value n_0 (see Ref. 28 for details), we obtain

$$\gamma=\left(\frac{n}{n_0}\right)^{-1/(\kappa-1/2)}-\frac{1}{(\kappa-1/2)}\left(\frac{n_0}{n}\right). \tag{10}$$

In the limit $\kappa\to\infty$, Eq. (10) shows that $\gamma\to 1$, that is, the isothermal regime is recovered in the framework of the Boltzmann–Gibbs statistics. We name the function γ, in Eq. (10), the first polytropic index.

Let us define the second polytropic index by

$$\Gamma=\frac{2}{f}+\gamma, \tag{11}$$

where f is the usual number of degrees of freedom of the system and γ is given by Eq. (10). In the limit $\kappa\to\infty$, Eq. (11) shows that

$$\Gamma\to\frac{2}{f}+1, \tag{12}$$

that is, the usual polytropic index is recovered (see Ref. 34). A word on the concept of polytropic index in the realm of classical statistics, in contrast with the new notions that arise from our proposal, is in order now. In the framework of Boltzmann–Gibbs statistics, the polytropic index, $\Gamma_{\rm BG}$, is formulated as the ratio of the specific heat at a constant pressure, c_P, to that at constant volume, c_V, namely,[34]

$$\Gamma_{\rm BG}=\frac{c_P}{c_V}=\frac{2+f}{f}, \tag{13}$$

where f is the aforementioned usual number of degrees of freedom of the system. Now, what we propose is the following: in the realm of non-Boltzmann–Gibbs statistics, c_P and c_V would conspire in such a way that f could not be the same in the numerator and denominator of Eq. (13). As a consequence, the ratio of those different numbers should yield γ, our first polytropic index, thereby leading Eq. (13) to coincide with Eq. (11), which defines Γ, our second polytropic index. Indeed, see Ref. 29 for the possibility of emergence of systems with non-integer numbers of degrees of freedom in the framework of non-Boltzmann–Gibbs statistics. We realize that such topics deserve further investigation, but, in this work, we focus our attention on a simple and important case in plasma physics, for which our ideas can be promptly checked: one-dimensional ($f=1$) Langmuir waves propagating in warm plasmas, while described by the Bohm–Gross dispersion relation.

Given the above considerations, we update our starting point by replacing Eq. (4) with

$$\nabla P=k_{\rm B}\Theta\nabla(\Gamma n), \tag{14}$$

where Θ and Γ are given by Eqs. (7) and (11), respectively. In the limit $\kappa\to\infty$, Eq. (14) shows that

$$\nabla P\to\left(\frac{2}{f}+1\right)k_{\rm B}T\nabla n, \tag{15}$$

that is, the usual adiabatic regime is recovered (see Ref. 34).

By substituting Eq. (10) in Eq. (11), we obtain

$$\Gamma=\frac{2}{f}+\left(\frac{n}{n_0}\right)^{-1/(\kappa-1/2)}-\frac{1}{(\kappa-1/2)}\left(\frac{n_0}{n}\right). \tag{16}$$

Then, by substituting Eq. (16) in Eq. (14), we integrate it to find out the equation of state of a gas of charged particles following the Kappa density distribution (6),

$$P=\left[\frac{2}{f}\left(\frac{n}{n_0}\right)+\left(\frac{n}{n_0}\right)^{(\kappa-3/2)/(\kappa-1/2)}-\frac{1}{(\kappa-1/2)}\right]\left(\frac{\kappa}{\kappa-3/2}\right)n_0k_{\rm B}T. \tag{17}$$

In the limit $\kappa\to\infty$, Eq. (17) shows that

$$P\to\left(\frac{2}{f}+1\right)nk_{\rm B}T, \tag{18}$$

that is, the adiabatic equation of state of a gas of charged particles following the Boltzmann relation is recovered (see Ref. 2).

III. BOHM–GROSS DISPERSION RELATION FOR KAPPA GAS

Consider the static state of equilibrium of the system, for which $\vec{v}=0$, $\Phi=0$, and $n=n_0$. Around such an equilibrium, regard the linear perturbation $\vec{v}\to\vec{v}_1$, $\Phi\to\Phi_1$, and $n\to n_0+n_1$.

As a result of the aforementioned disturbance, Eq. (1) yields

$$\frac{\partial\vec{v}_1}{\partial t}=\left(\frac{e}{m}\right)\nabla\Phi_1-\left[\frac{2}{f}\left(\frac{\kappa}{\kappa-3/2}\right)+\left(\frac{\kappa}{\kappa-1/2}\right)\right]\left(\frac{k_{\rm B}T}{m}\right)\nabla\left(\frac{n_1}{n_0}\right), \tag{19}$$

where use has been made of Eq. (17) in the limit $n\to n_0+n_1$. In the same approximation, the continuity and Poisson equations imply

$$\frac{\partial}{\partial t}\left(\frac{n_1}{n_0}\right)=-\nabla\cdot\vec{v}_1,\qquad\nabla^2\Phi_1=\left(\frac{n_0e}{\epsilon_0}\right)\left(\frac{n_1}{n_0}\right), \tag{20}$$

respectively, where ϵ_0 is the vacuum electric permittivity. By combining Eq. (19) with both Eq. (20), we obtain

$$-\frac{\partial^2}{\partial t^2}\left(\frac{n_1}{n_0}\right)=\omega_{\rm pe}^2\left(\frac{n_1}{n_0}\right)-\left[\frac{2}{f}\left(\frac{\kappa}{\kappa-3/2}\right)+\left(\frac{\kappa}{\kappa-1/2}\right)\right]v_{\rm th}^2\nabla^2\left(\frac{n_1}{n_0}\right), \tag{21}$$

where we have introduced the abbreviations

$$\omega_{\rm pe}=\left(\frac{n_0e^2}{\epsilon_0m}\right)^{1/2},\quad v_{\rm th}=\left(\frac{k_{\rm B}T}{m}\right)^{1/2} \tag{22}$$

for the electron plasma frequency and isothermal speed, respectively.[2]

Let us assume a sinusoidal-like functional dependence for the perturbation

$$\frac{n_1}{n_0}\sim\exp\left(\imath\vec{k}\cdot\vec{r}-\imath\omega t\right), \tag{23}$$

where ω and $\vec{k}$ are the angular frequency and wave vector, respectively. On the account of expression (23) in Eq. (21), we find the dispersion relation of the frequency ω with the wavenumber $k=|\vec{k}|$,

$$\frac{\omega^2}{\omega_{\rm pe}^2} = 1 + \left[\frac{2}{f}\left(\frac{\kappa}{\kappa - 3/2}\right) + \left(\frac{\kappa}{\kappa - 1/2}\right)\right] k^2 \lambda_{\rm De}^2, \tag{24}$$

where we have introduced the abbreviation

$$\lambda_{\rm De} = \frac{v_{\rm th}}{\omega_{\rm pe}} = \left(\frac{\epsilon_0 k_{\rm B} T}{n_0 e^2}\right)^{1/2} \tag{25}$$

for the electron Debye length.[2] It should be noticed that limiting properties of Eq. (24) have been studied in the framework of kinetic theory.[35–37] However, to the best of our knowledge, this is the first time that it is derived in its closed form with no resort to an approximate method. In the limit $\kappa \to \infty$, Eq. (24) approaches

$$\frac{\omega^2}{\omega_{\rm pe}^2} = 1 + \left[\frac{2}{f} + 1\right] k^2 \lambda_{\rm De}^2. \tag{26}$$

Then, for $f = 1$, Eq. (26) recovers the Bohm–Gross dispersion relation in the framework of Boltzmann–Gibbs statistics,[6]

$$\frac{\omega^2}{\omega_{\rm pe}^2} = 1 + 3k^2 \lambda_{\rm De}^2, \tag{27}$$

which describes the propagation of one-dimensional Langmuir waves in warm plasmas.[1] A last, but not least, remark is in order now.

Equation (24) may be recast in a form similar to that of Eq. (26), namely,

$$\frac{\omega^2}{\omega_{\rm pe}^2} = 1 + \left[\frac{2}{F} + 1\right] k^2 \Lambda_{\rm De}^2, \tag{28}$$

where we have introduced the abbreviations

$$F = \left(\frac{\kappa - 3/2}{\kappa - 1/2}\right) f, \quad \Lambda_{\rm De} = \left(\frac{\kappa - 3/2}{\kappa - 1/2}\right)^{1/2} \frac{V_{\rm th}}{\omega_{\rm pe}} \tag{29}$$

for the κ-number F of degrees of freedom of the system and electron Debye κ-length $\Lambda_{\rm De}$, respectively, the latter expressed in terms of

$$V_{\rm th} = \left(\frac{k_{\rm B}\Theta}{m}\right)^{1/2}, \tag{30}$$

the electron isothermal κ-speed $V_{\rm th}$, with the Kappa Θ-temperature given by Eq. (7). Several studies on modifications of the Debye length have been pursued in plasma physics. For instance, in Ref. 38, the problem has been examined for a one-electron component non-Maxwellian plasma. Equation 20 of Ref. 38 coincides with the second of our Eq. (29). Modified Debye lengths have been also obtained for hot and cold electrons of a non-Maxwellian plasma in Ref. 39, which includes numerical treatments of the consequent Debye shielding. We think that our analytical formulation could be directly extended to two-particle plasma components, and then its results could be compared with those of Ref. 39. Now, an interesting investigation has been performed by Livadiotis,[40] who formulated a generalized Debye length in terms of extended (named correlated) numbers of degrees of freedom of a non-Maxwellian plasma. It is true that some effort must be made to compare our results with those of Ref. 40, given that the Livadiotis theory is based on a Hamiltonian formulation. This issue will be explored in future communications.

IV. VAN HOVEN AND DERFLER–SIMONEN EXPERIMENTS

In 1966, van Hoven[7] and Derfler and Simonen[8] observed, in an independent way, the propagation of one-dimensional electrostatic plane waves in non-collisional thermal plasmas in the absence of a magnetic field. At that time, there was already a theoretical prediction of electrostatic-wave propagation, established by Bohm and Gross.[6] However, no experimental verification of that had been yet carried out in the laboratory. The experiments intended to fully validate the ω–k relation while predicted by Eq. (27). The main idea was to apply an input signal at various frequencies, which excited electrostatic waves in the plasma and then measure the wave numbers of the received signal.

According to the technical specifications of each experiment, some physical parameters were evaluated. For the van Hoven apparatus, the electron equilibrium concentration was estimated to be $n_0 \sim 5 \times 10^7\,{\rm cm}^{-3}$, linear plasma frequency, $\nu_{\rm pe} \sim 59\,{\rm MHz}$, and Debye length, $\lambda_{\rm De} \sim 1\,{\rm mm}$. For the Derfler and Simonen experiment, $n_0 \sim 2 \times 10^7\,{\rm cm}^{-3}$, $\nu_{\rm pe} \sim 34\,{\rm MHz}$, and $\lambda_{\rm De} \sim 0.7\,{\rm mm}$. The values of $2\pi\nu_{\rm pe}$ were used to normalize the angular frequencies ω, and those of $\lambda_{\rm De}$, the wavenumbers k.

The results presented by van Hoven and Derfler–Simonen exhibited all the basic features of electrostatic plasma waves. However, as we will show soon, they are far from reasonable agreement with the Bohm–Gross dispersion relation. The reason for that is the following.

In data analysis, the method of least squares provides a measure of how much a given fit line departs from the data distribution, by selecting the regression equation that minimizes the so-called sum of squared residuals, $S_{\rm res}$. Such a quantity is defined as the sum of squared differences between each one of the observed values of the dependent variable, Y_i, and the corresponding value yield by the regression equation, $G(X_i)$, calculated at the associated observed value of the independent variable, X_i, where $i = 1, 2, \ldots, N$ labels each one of the N data. In the present case, one seeks the values of A and B, which minimize[41]

$$S_{\rm res} = \sum_i [Y_i - G(X_i)]^2 = \sum_i \left[Y_i - \left(B + AX_i^2\right)^{1/2}\right]^2, \tag{31}$$

where the independent X_i and dependent Y_i variables are defined by

$$X_i = k_i \lambda_{\rm De}, \quad Y_i = \frac{\omega_i}{\omega_{\rm pe}} \tag{32}$$

for each one of the observed wavenumber k_i and frequency ω_i.

Now, by considering the mean value of the observed data,

$$\bar{Y} = \frac{1}{N}\sum_i Y_i, \tag{33}$$

one can define the quantity[41]

$$S_{\rm tot} = \sum_i [Y_i - \bar{Y}]^2, \tag{34}$$

a measure of the total variation of the dependent variables with respect to its mean value.

Finally, one can define the so-called coefficient of determination, R^2, through[41]

$$R^2 = 1 - \frac{S_{\rm res}}{S_{\rm tot}}. \tag{35}$$

Such a quantity indicates how well a given regression equation describes the relationship between the two observed variables. On the one hand, a result $R^2 = 1$ specifies that the dependent variable can be predicted with certainty from the independent one, once there is no residual variation around the regression equation, $S_{res} = 0$. On the other hand, an outcome $R^2 \leq 0$ stipulates that no regression trend stems from the observed data, since the residual variation around the regression equation is greater or equal to the total variation around the mean value of the dependent variable, $S_{res} \geq S_{tot}$.

The above explained regression method of least squares is directly applicable to the Bohm–Gross dispersion relation, Eq. (27), derived in the framework of Boltzmann–Gibbs statistics. However, in order to apply it in the realm of Kappa statistics, we must, first, choose $f = 1$ in our Eq. (24), thereby obtaining

$$\frac{\omega^2}{\omega_{pe}^2} = 1 + \left[2\left(\frac{\kappa}{\kappa - 3/2}\right) + \left(\frac{\kappa}{\kappa - 1/2}\right)\right] k^2 \lambda_{De}^2. \tag{36}$$

Then, the coefficient A in Eq. (31) will be expressed in terms of the spectral κ-index through

$$A = 2\left(\frac{\kappa}{\kappa - 3/2}\right) + \left(\frac{\kappa}{\kappa - 1/2}\right) \tag{37}$$

to be properly determined by the regression method in the interval $3/2 < \kappa < \infty$.

In Figs. 1 and 2, we show the results of the aforementioned application to fit the van Hoven and Derfler–Simonen data, respectively, by Eq. (36), in contrast with Eq. (27). We find out that $\kappa = 3.03 \pm 0.11$ and $\kappa = 2.03 \pm 0.06$, with coefficients of determination $R^2 = 0.94$ and $R^2 = 0.85$, for the fit of the van Hoven and Derfler–Simonen data, respectively, by Eq. (36). Therefore, in view of the above discussion, we conclude that such results are, themselves, extremely satisfactory, even more, by taking into account $R^2 = 0.52$ and $R^2 = -0.10$, for the fit of the van Hoven and Derfler–Simonen data, respectively, by Eq. (27).

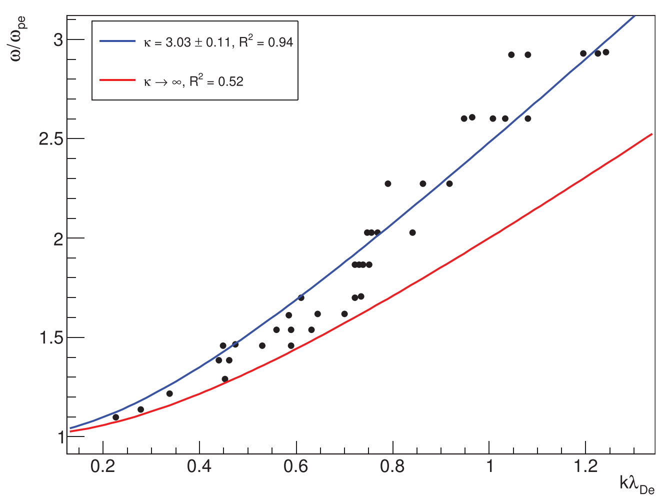

FIG. 1. The data distribution (circles on the plane) for the wavenumber k and frequency ω normalized by the electron Debye length λ_{De} and plasma frequency ω_{pe}, as observed in the experiment conducted by van Hoven.[7] The regression method of least squares[41] is applied to fit the data distribution by the suprathermal Bohm–Gross dispersion relation, Eq. (36) (the curve to the top of the plane, blue online), in contrast with the usual Bohm–Gross dispersion relation, Eq. (27) (the curve to the bottom of the plane, red online). We find out that the spectral index $\kappa = 3.03 \pm 0.11$, with coefficient of determination $R^2 = 0.94$, for the fit by Eq. (36). Such results are, themselves, extremely satisfactory, even more, by taking into account $R^2 = 0.52$, for the fit by Eq. (27).

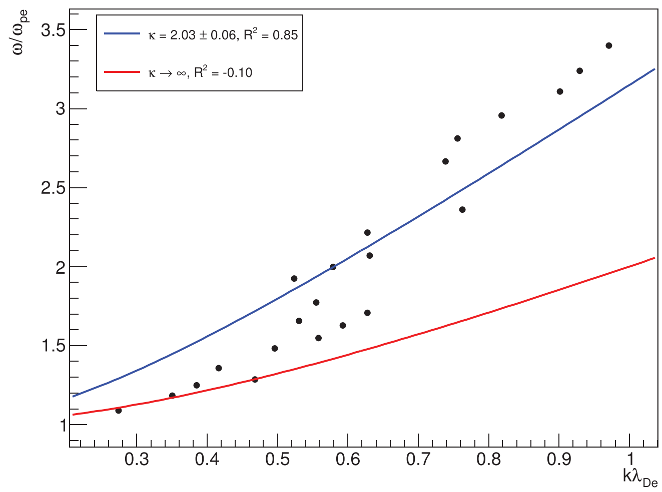

FIG. 2. The data distribution (circles on the plane) for the wavenumber k and frequency ω normalized by the electron Debye length λ_{De} and plasma frequency ω_{pe}, as observed in the experiment conducted by Derfler and Simonen.[8] The regression method of least squares[41] is applied to fit the data distribution by the suprathermal Bohm–Gross dispersion relation, Eq. (36) (the curve to the top of the plane, blue online), in contrast with the usual Bohm–Gross dispersion relation, Eq. (27) (the curve to the bottom of the plane, red online). We find out that the spectral index $\kappa = 2.03 \pm 0.06$, with coefficient of determination $R^2 = 0.85$, for the fit by Eq. (36). Such results are, themselves, extremely satisfactory, even more, by taking into account $R^2 = -0.10$, for the fit by Eq. (27).

V. CONCLUSION

We have proved that the employment of our derived suprathermal corrections to the standard Bohm–Gross dispersion has been in excellent agreement with the van Hoven and Derfler–Simonen experimental data. Our derivation has been based on an extension of the usual notion of a polytropic index to non-Boltzmann–Gibbs statistics. Such an extension has allowed for the deduction of an equation of state of charged particles with basis on the Kappa distribution. Let us mention a handful of possible applications of our theory.

The breaking of Langmuir waves, propagating with phase speed v_{ph} in a plasma of warm electrons and cold ions, is known to be limited by the criterion,[42]

$$\left(\frac{n}{n_0}\right)^{\Gamma_{BG}+1} \leq \frac{v_{ph}^2}{\Gamma_{BG}^2 v_{th}^2}, \tag{38}$$

where Γ_{BG} is the Boltzmann–Gibbs adiabatic index, while formulated as Eq. (13). It shall be interesting to investigate the consequences of expressing inequality (38) in terms of our second polytropic index Γ, while defined by Eq. (11). Actually, Langmuir wave-breaking is of great importance in several processes occurring in plasma physics, such as laser pulse compression to extremely short duration[43] and linear–nonlinear mode transition driven by phase synchronization.[44]

In Boltzmann–Gibbs statistics, the nonlinear electric permittivity, ϵ, in a plasma of warm electrons and cold ions, may be expressed through[45]

$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2}\exp\left(-\frac{e^2\langle E^2\rangle}{2m^2 v_{\rm th}^2\omega^2}\right), \tag{39}$$

where $\langle\cdot\rangle$ stands for time averaging and $\vec{E}$ denotes the so-called ponderomotive electric field, oscillating at an angular frequency ω. It shall be interesting to examine how Eq. (39) modifies in view of our theory because ϵ plays a key role in inertial confinement fusion plasmas.[46] As a matter of fact, acceleration of electrons from regions with concentration $\sim$1% of the critical concentration, driven by laser irradiance $>10^{16}$ W cm^{-2} μm^2, has been recently achieved.[47]

It is well-known that the ponderomotive field in Eq. (39) causes a density fluctuation in a warm electron fluid, which, in turn, gives rise to the so-called nonlinear Bohm–Gross dispersion relation,[2]

$$\frac{\omega^2}{\omega_{\rm pe}^2} = 1 + \frac{\delta n}{n_0} + 3k^2\lambda_{\rm De}^2, \tag{40}$$

where δn stands for a second-order disturbance in the equilibrium concentration. It shall be interesting to explore how Eq. (40) modifies in light of our theory, given that density fluctuations are crucial to describing radio emission from interplanetary shocks, planetary foreshocks, and some solar flares, a plethora of phenomena commonly referred to as the plasma emission framework.[48–50] All those above-mentioned issues shall be addressed in forthcoming communications.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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