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ABSTRACT

We provide a formulation that describes the propagation of solitons in a nondissipative, nonmagnetic plasma, which does not depend on the particular electron density distribution considered. The Poisson equation in the plasma sheath is expressed in terms of the Mach number for ions entering the sheath from the plasma and of a natural scale for the electrostatic potential. We find a class of reference frames with respect to which certain functions become stationary after arbitrary small variations of the Mach number and potential scale, that is, by determining the critical values of those quantities based on a variational method. It is shown that the critical Mach number defines the limits for the applicability of the reductive perturbation technique to a given electron density distribution. Based on our provided potential scale, we show that the Taylor expansion of the suprathermal electron distribution around equilibrium converges for all possible values of the spectral κ-index. In addition, owing to the admissible range for the critical Mach number, it is found that the reductive perturbation technique ceases to be valid for $3/2 < \kappa \leq 5/2$. In the sequel, we show that the technique is not valid for the deformation q-index of nonextensive electrons when $q \leq 3/5$. Furthermore, by assuming that the suprathermal and nonextensive solitons are both described with respect to the same critical reference frame, a relation between κ and q, which has been previously obtained on very fundamental grounds, is recovered.

I. INTRODUCTION

Soliton (or solitary) waves (or oscillations) are spatially localized structures traveling in a way that nonlinearity balances out dispersion. Investigations of solitons find countless applications in diverse areas, ranging from fiber optics[1] to neurobiology,[2] nuclear physics,[3] and even two-dimensional quantum systems.[4] Modern soliton theory owes much to the seminal work of Gardner, Greene, Kruskal, and Miura.[5] Following that formulation, many interesting predicted configurations, such as high-order polarization-locked vector solitons[6] and accelerating solitary wavepackets,[7] have been recently observed.

In a Boltzmann plasma, the Korteweg–de Vries equation that describes the propagation of nonlinear, dispersive, small-amplitude ion-acoustic waves may be derived by making use of the reductive perturbation technique.[8,9] Based on that approach, suprathermal electron effects on envelope solitons,[10] dust-acoustic and dust-ion-acoustic shock waves,[11] and multi-ion dusty plasmas in the presence of a magnetic field[12] have been investigated. Nonextensive influences on ion-acoustic solitons following the Tsallis[13] and Cairns-Tsallis models,[14] and dusty plasmas[15] have been also examined on the same ground. Even Majorana solitons,[16] soliton trains,[17] and solitonic cascades[18] in a Fermi gas have been recently explored on that foundation.

It should be noticed, however, that the applicability of the reductive perturbation may be limited to certain values of the relevant physical parameters. One of those instances has attracted our attention. In Refs. 19 and 20, it has been claimed that the technique is not valid for $3/2 < \kappa \leq 3$, given that, in general, $\kappa > 3/2$, where κ is the spectral index marking suprathermal electrons. The authors have credited their result to the fact that the Taylor expansion of the electron distribution around equilibrium diverges for $3/2 < \kappa \leq 3$. Nonetheless, it should be emphasized that such a justification heavily relies on the choice of the normalization for the electrostatic potential. Those authors simply have chosen the usual Boltzmann thermal energy per elementary charge as the normalization. So, what if we could choose another normalization? In that case, would the reductive perturbation remain invalid for the same range of the spectral index? That has been the original motivation for this work.

This paper is organized as follows. In Sec. II, we express the Poisson equation in the plasma sheath in terms of the Mach number for ions entering the sheath from the plasma and of a natural scale for the electrostatic potential. In Sec. III, we find a class of reference frames with respect to which certain functions become stationary after arbitrary small variations of the Mach number and potential scale, that is, by determining the critical values of those quantities based on a variational method. In Sec. IV, we show that it is actually the critical Mach number that defines the limits for the applicability of the reductive perturbation technique to a given electron density distribution.

In Sec. V, we apply our proposed formulation to four different electron distributions. First, we recover the usual results for the Boltzmann distribution. Then, based on our provided potential scale, we show that the Taylor expansion of the suprathermal electron distribution around equilibrium converges for all possible values of the spectral κ-index. In addition, owing to the admissible range for the critical Mach number, we find that the reductive perturbation technique ceases to be valid for $3/2 < \kappa \leq 5/2$.

In the sequel, we show novel results for the deformation q-index of nonextensive electrons, similar to those for κ. Furthermore, by assuming that the suprathermal and nonextensive solitons are both described with respect to the same critical reference frame, we recover a relation between κ and q which has been previously obtained on very fundamental grounds. Finally, we find that our model may be applied even to a Fermi gas. Moreover, we offer independent confirmation of that. In the concluding section, we summarize this work and briefly address possible extensions of our theory.

II. MACH NUMBER AT PLASMA-SHEATH INTERFACE

In an ionized gas, the layer surrounding the plasma is dubbed the sheath. Different from the plasma, in which the densities of ions and electrons are high, but essentially equal, the number of ions much exceeds that of electrons in the sheath. Let us consider singly ionized ions with charge $e > 0$, mass m_i, concentration N_0, velocity $\vec{V}_0$, and negligible temperature, entering the sheath from the plasma. As usual, the electrostatic potential Φ is assumed to decrease very rapidly with the distance from the plasma-sheath interface to the ionized gas external wall. Since we regard a steady state problem in a collisionless region, dissipative processes may be ignored, and the conservations of mass and energy of the ion gas may be expressed through

$$N_0 \vec{V}_0 = N_i \vec{V}_i, \quad \frac{m_i V_0^2}{2} = \frac{m_i V_i^2}{2} + e\Phi, \tag{1}$$

respectively, where N_i and $\vec{V}_i$ are the concentration and velocity, respectively, of ions within the sheath. Combining both Eq. (1), we get

$$\frac{N_i}{N_0} = \left(1 - \frac{2e\Phi}{m_i V_0^2}\right)^{-1/2}. \tag{2}$$

Equation (2) may be then interpreted as a distribution-like function.

By including the concentration of electrons N_e in the problem, the potential profile in the plasma sheath is controlled by the Poisson equation,

$$\nabla^2\Phi = -\frac{e}{\epsilon_0}\{N_i - N_e\}, \tag{3}$$

where ϵ_0 is the vacuum electric permittivity. Both particle concentrations in Eq. (3) may be expressed in terms of the equilibrium concentration N_0 through

$$N_i = N_0 F_i(\Phi), \quad N_e = N_0 F_e(\Phi), \tag{4}$$

where $F_i(\Phi)$ is given by the right-hand side of Eq. (2) and $F_e(\Phi)$ is treated as an abstract function of the potential. Substituting Eq. (4) in Eq. (3), we have

$$\nabla^2\Phi = -\frac{N_0 e}{\epsilon_0}\{F_i(\Phi) - F_e(\Phi)\}. \tag{5}$$

Let us Taylor-expand $F_i(\Phi)$ and $F_e(\Phi)$ around their equilibrium values $F_i(0) = F_e(0) = 1$,

$$F_i(\Phi) \approx 1 + \frac{e\Phi}{m_i V_0^2} + \frac{3e^2\Phi^2}{2m_i^2 V_0^4}, \quad F_e(\Phi) \approx 1 + F_e'(0)\Phi + \frac{F_e''(0)\Phi^2}{2}, \tag{6}$$

where a prime denotes the derivative with respect to Φ. Substituting Eq. (6) in Eq. (5), we obtain

$$\nabla^2\Phi = -\frac{N_0 e}{\epsilon_0}\left\{\left[\frac{e}{m_i V_0^2} - F_e'(0)\right]\Phi + \frac{1}{2}\left[\frac{3e^2}{m_i^2 V_0^4} - F_e''(0)\right]\Phi^2\right\}. \tag{7}$$

We now express the ion flow intensity V_0 at the plasma-sheath interface in a more convenient form for our purposes.

The ratio of the intensity of the flow past a fluid boundary to the local speed of sound is termed the Mach number. The natural timescale of an ion-acoustic oscillation is given by the inverse of the ion-plasma frequency,

$$\omega_i = \left(\frac{N_0 e^2}{\epsilon_0 m_i}\right)^{1/2}. \tag{8}$$

Introducing the length scale λ_e (the meaning of the index "e" included in our notation will be further clarified) in the problem and subsequently requiring that the speed of sound may be expressed through $\lambda_e\omega_i$, we may define the Mach number for the ion flow at the plasma-sheath interface through

$$\mathcal{M} = \frac{V_0}{\lambda_e \omega_i}. \tag{9}$$

Substituting Eq. (8) in Eq. (9), we get

$$\lambda_e = \left(\frac{\epsilon_0}{N_0 e}\right)^{1/2}\left(\frac{m_i V_0^2}{e\mathcal{M}^2}\right)^{1/2}. \tag{10}$$

Now, since λ_e is our length scale, the dimensional analysis of Eq. (5) shows that

$$\lambda_e = \left(\frac{\epsilon_0}{N_0 e}\right)^{1/2}\varphi^{1/2}, \tag{11}$$

where φ is the natural potential scale. Therefore, comparing Eqs. (10) and (11), we have

$$\frac{e}{m_i V_0^2} = \frac{1}{\varphi\mathcal{M}^2}. \tag{12}$$

Finally, substituting Eq. (12) in Eq. (7), we obtain

$$\nabla^2\Phi = -\frac{N_0 e}{\epsilon_0}\left\{\left[\frac{1}{\varphi\mathcal{M}^2} - F_e'(0)\right]\Phi + \frac{1}{2}\left[\frac{3}{\varphi^2\mathcal{M}^4} - F_e''(0)\right]\Phi^2\right\}. \tag{13}$$

In Sec. III, we show that Eq. (13) may be used to determine a natural potential scaling in our problem.

III. POTENTIAL SCALING BY VARIATIONAL METHOD

Let us suppose that the potential scale in our problem is determined by requiring that φ satisfies the condition,

$$\left[\frac{1}{\varphi\mathcal{M}^2} - F_e'(0)\right] = \frac{\varphi}{2}\left[\frac{3}{\varphi^2\mathcal{M}^4} - F_e''(0)\right], \tag{14}$$

which is dimensionally consistent with Eq. (13). Our hypothesis expressed through Eq. (14) may be manifested as the following variational problem. We seek the critical value of $\mathcal{M}^2$ that keeps the function,

$$\mathcal{F}(\mathcal{M}^2) = \left[F_e''(0)\varphi - 2F_e'(0)\right]\varphi\mathcal{M}^4 + 2\mathcal{M}^2 - 3 = 0, \tag{15}$$

stationary after an arbitrary small variation of the Mach number. To solve the problem, we differentiate Eq. (15) with respect to $\mathcal{M}^2$ to get

$$\mathcal{F}'(\mathcal{M}^2) = 2\left[F_e''(0)\varphi - 2F_e'(0)\right]\varphi\mathcal{M}^2 + 2 = 0. \tag{16}$$

The solution of Eq. (16) is given by

$$\frac{1}{\varphi\mathcal{M}^2} = 2F_e'(0) - F_e''(0)\varphi. \tag{17}$$

Equation (17) states the following. Among all possible values of $\mathcal{M}^2$ that satisfy Eq. (15), those given by Eq. (17) keep Eq. (15) stationary after an arbitrary small variation of the Mach number. Therefore, we have not determined the value of the Mach number for the ion flow at the plasma-sheath interface. What we have determined was the critical value of the Mach number that defines a class of reference frames with respect to which ion-acoustic waves may be described in our problem. We are not yet done.

To determine the potential scale properly, we substitute Eq. (17) in Eq. (14), which leads us to a further variational problem, that of to seek the critical value of φ which keeps the function,

$$f(\varphi) = 3F_e''(0)\varphi^2 - 6F_e'(0)\varphi + 1 = 0, \tag{18}$$

stationary after an arbitrary small variation of the potential scale. We differentiate Eq. (18) with respect to φ to get

$$f'(\varphi) = 6F_e''(0)\varphi - 6F_e'(0) = 0. \tag{19}$$

The solution of Eq. (19) is given by

$$\varphi = \frac{F_e'(0)}{F_e''(0)}. \tag{20}$$

Among all possible values of the potential scale that satisfy Eq. (18), those given by Eq. (20) keep Eq. (18) stationary after an arbitrary small variation of φ. We are done now.

It follows from Eq. (20) that the critical length scale, ion flow, and Mach number, all may be fully expressed in terms of just the first and second derivatives of the electron distribution with respect to the electrostatic potential at equilibrium through

$$\lambda_e = \left(\frac{\epsilon_0}{N_0 e}\right)^{1/2}\left[\frac{F_e'(0)}{F_e''(0)}\right]^{1/2}, \quad V_0 = \left[\frac{e/m_i}{F_e'(0)}\right]^{1/2}, \quad \mathcal{M} = \left\{\frac{F_e''(0)}{\left[F_e'(0)\right]^2}\right\}^{1/2}, \tag{21}$$

respectively. In Sec. IV, we show that the last of Eq. (21) gives the coefficients of the nonlinear and dispersive terms of the Korteweg–de Vries equation in the plasma bulk.

IV. KORTEWEG–DE VRIES EQ.

A. Ion and electron-acoustic waves, and shock structures

The Korteweg–de Vries equation in one space dimension may be expressed through

$$\frac{\partial\psi}{\partial\theta} + B\psi\frac{\partial\psi}{\partial\chi} + \frac{C}{2}\frac{\partial^3\psi}{\partial\chi^3} = 0, \tag{22}$$

where ψ is some low-amplitude field, θ and χ are dimensionless time and space coordinates, respectively, and B and C are constants. Equation (22) allows for wave-like solutions. Introducing the phase speed β, we may define the wave-front coordinate $\sigma = \chi - \beta\theta$. Hence, on the usual assumption that ψ and its derivatives, all become vanishingly small in the limit $\sigma \to \pm\infty$, we may integrate Eq. (22) twice to get

$$\left(\frac{d\psi}{d\sigma}\right)^2 = \frac{2\beta}{C}\psi^2 - \frac{2B}{3C}\psi^3, \tag{23}$$

with the help of $d\psi/d\sigma$ as an integrating factor. The soliton wave then comes about as the solution of Eq. (23), namely,

$$\psi(\sigma) = \frac{3\beta}{B}\operatorname{sech}^2\left[\sigma\left(\frac{\beta}{2C}\right)^{1/2}\right]. \tag{24}$$

In this work, we will derive Eq. (22) in the framework of our proposed analytical formulation. Accordingly, a slight modification will be introduced in the usual reductive perturbation technique. Our starting point will be the ion fluid and Poisson equations. Thus, Eq. (24) will describe ion-acoustic waves. A couple of comments on possible descriptions of other wave-like structures within the realm of our formulation are in order now.

The so-called electron-acoustic waves were numerically explored by Fried and Gould[21] in uniform and unmagnetized plasmas. Later on, analytic studies by Watanabe and Taniuti[22] showed that such structures may be actually interpreted as acoustic modes, provided that the thermal pressure due to the hotter electrons be responsible for the restoring force, and inertial effects be attributable to the colder electrons. The electron-acoustic waves have been intensively investigated in both space[23–26] and laboratory[27–32] plasmas. Since the electron mass m_e is much lower than the ion mass m_i, the electron-plasma frequency ω_e will be much higher than the ion-plasma frequency ω_i,

$$\omega_e = \omega_i\left(\frac{m_i}{m_e}\right)^{1/2}, \tag{25}$$

where ω_i is given by Eq. (8). Therefore, we think that our proposed formulation could be applicable to the description of the electron-acoustic waves whether the characteristic timescale in the problem would be determined by the inverse of the electron-plasma frequency for the colder electrons [cf. the second of Eq. (28) below].

Now, when the usual assumption of a vanishingly small field at infinity is not appropriate, Eq. (23) will be replaced with

$$\left(\frac{d\psi}{d\sigma}\right)^2 = -\frac{4C_0}{C} - \frac{4C_1}{C}\psi + \frac{2\beta}{C}\psi^2 - \frac{2B}{3C}\psi^3, \tag{26}$$

where C_0 and C_1 are constants of integration. The general solution of Eq. (26) may be expressed in terms of the well-known periodic elliptic (or Jacobi) functions.[33] In particular, the application of the so-called Whitham modulation technique to the small dispersion limit of Eq. (26) leads to shock-like solutions.[34–36] Consequently, we think that our proposed formulation cannot be applied at once to the description of shock structures, otherwise the relation of ψ with further relevant small-amplitude fields in the problem may become unsatisfactorily involved [cf. Eq. (37) below].

B. Modified reductive perturbation technique

In Secs. II and III, we have analyzed the physics in the plasma sheath in order to determine natural scalings of relevant quantities in our problem. We now move to the plasma bulk. Consider the simple case of motions in one space dimension. Since the ion temperature may be neglected, the fluid equations of continuity and motion, and Poisson equation are

$$\frac{\partial N_i}{\partial t} = -\frac{\partial}{\partial x}(N_i V_i), \quad \frac{\partial V_i}{\partial t} + V_i\frac{\partial V_i}{\partial x} = -\frac{e}{m_i}\frac{\partial \Phi}{\partial x}, \quad \frac{\partial^2\Phi}{\partial x^2} = -\frac{e}{\epsilon_0}(N_i - N_e), \tag{27}$$

respectively. The dimensionless variables

$$\xi = \frac{x}{\lambda_e}, \quad \tau = \omega_i t, \quad n_i = \frac{N_i}{N_0}, \quad n_e = \frac{N_e}{N_0}, \quad v_i = \frac{V_i}{\lambda_e\omega_i}, \quad \phi = \frac{\Phi}{\varphi} \tag{28}$$

will make all the coefficients unity in Eq. (27),

$$\frac{\partial n_i}{\partial \tau} = -\frac{\partial}{\partial \xi}(n_i v_i), \quad \frac{\partial v_i}{\partial \tau} + v_i\frac{\partial v_i}{\partial \xi} = -\frac{\partial \phi}{\partial \xi}, \quad \frac{\partial^2 \phi}{\partial \xi^2} = -(n_i - n_e). \tag{29}$$

As usual, we regard the expansions around equilibrium,[37,38]

$$n_i = 1 + \delta n_1 + \delta^2 n_2, \quad v_i = \delta v_1 + \delta^2 v_2, \quad \phi = \delta\phi_1 + \delta^2\phi_2, \tag{30}$$

where $\delta = |\mathcal{M} - 1| \ll 1$. Following the last of Eq. (6), the expansion of the electron distribution around equilibrium becomes

$$n_e = 1 + \delta\frac{\phi_1}{\mathcal{M}^2} + \delta^2\left(\frac{\phi_2}{\mathcal{M}^2} + \frac{\phi_1^2}{2\mathcal{M}^2}\right). \tag{31}$$

Use of Eqs. (30) and (31) turns Eq. (29) into

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(\delta n_1 + \delta^2 n_2) = -\frac{\partial}{\partial \xi}[\delta v_1 + \delta^2(v_2 + n_1 v_1)],\\
&\frac{\partial}{\partial \tau}(\delta v_1 + \delta^2 v_2) + \frac{\partial}{\partial \xi}\left(\frac{\delta^2 v_1^2}{2}\right) = -\frac{\partial}{\partial \xi}(\delta\phi_1 + \delta^2\phi_2),\\
&\frac{\partial^2}{\partial \xi^2}(\delta\phi_1 + \delta^2\phi_2) = -\delta\left(n_1 - \frac{\phi_1}{\mathcal{M}^2}\right) - \delta^2\left(n_2 - \frac{\phi_2}{\mathcal{M}^2} - \frac{\phi_1^2}{2\mathcal{M}^2}\right),
\end{aligned} \tag{32}$$

respectively. We introduce now a slight modification in the usual reductive perturbation technique.[8,9]

Let us define the stretched variables ζ and η in a reference frame moving with the critical velocity $v_i = \mathcal{M}$ with respect to the laboratory,

$$\zeta = \delta^{1/2}(\xi - \mathcal{M}\tau), \quad \eta = \delta^{3/2}\tau. \tag{33}$$

As a consequence of Eq. (33), the well-known chain rule of partial derivatives provides

$$\frac{\partial}{\partial \xi} = \delta^{1/2}\frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial \tau} = -\delta^{1/2}\mathcal{M}\frac{\partial}{\partial \zeta} + \delta^{3/2}\frac{\partial}{\partial \eta}. \tag{34}$$

Use of Eq. (34) turns Eq. (32) into

$$\begin{aligned}
&\delta^{3/2}\mathcal{M}\frac{\partial n_1}{\partial \zeta} + \delta^{5/2}\left(\mathcal{M}\frac{\partial n_2}{\partial \zeta} - \frac{\partial n_1}{\partial \eta}\right) = \delta^{3/2}\frac{\partial v_1}{\partial \zeta} + \delta^{5/2}\frac{\partial}{\partial \zeta}(v_2 + n_1 v_1),\\
&\delta^{3/2}\mathcal{M}\frac{\partial v_1}{\partial \zeta} + \delta^{5/2}\left(\mathcal{M}\frac{\partial v_2}{\partial \zeta} - \frac{\partial v_1}{\partial \eta} - v_1\frac{\partial v_1}{\partial \zeta}\right) = \delta^{3/2}\frac{\partial \phi_1}{\partial \zeta} + \delta^{5/2}\frac{\partial \phi_2}{\partial \zeta},\\
&\delta^2\frac{\partial^2\phi_1}{\partial \zeta^2} = -\delta\left(n_1 - \frac{\phi_1}{\mathcal{M}^2}\right) - \delta^2\left(n_2 - \frac{\phi_2}{\mathcal{M}^2} - \frac{\phi_1^2}{2\mathcal{M}^2}\right),
\end{aligned} \tag{35}$$

respectively. The terms of order $\delta^{3/2}$ in the first, and second of Eq. (35), and the term of order δ in the last of Eq. (35) lead to

$$\mathcal{M}\frac{\partial n_1}{\partial \zeta} = \frac{\partial v_1}{\partial \zeta}, \quad \mathcal{M}\frac{\partial v_1}{\partial \zeta} = \frac{\partial \phi_1}{\partial \zeta}, \quad n_1 = \frac{\phi_1}{\mathcal{M}^2}, \tag{36}$$

respectively. On the usual assumption that $n_1, v_1, \phi_1 \to 0$ in the limit $\zeta \to \pm\infty$, the integration of the two first of Eq. (36) shows that the first-order amplitudes become related to each other through powers of the critical Mach number,

$$n_1 = \frac{v_1}{\mathcal{M}} = \frac{\phi_1}{\mathcal{M}^2}, \tag{37}$$

which, as expected, is consistent with the last of Eq. (36). Now, the use of Eq. (37) reduces Eq. (35) to

$$\begin{aligned}
&-\frac{\mathcal{M}^3}{2}\frac{\partial n_2}{\partial \zeta} + \frac{1}{2}\frac{\partial \phi_1}{\partial \eta} = -\frac{\mathcal{M}^2}{2}\frac{\partial v_2}{\partial \zeta} - \frac{1}{\mathcal{M}}\phi_1\frac{\partial \phi_1}{\partial \zeta},\\
&-\frac{\mathcal{M}^2}{2}\frac{\partial v_2}{\partial \zeta} + \frac{1}{2}\frac{\partial \phi_1}{\partial \eta} + \frac{1}{2\mathcal{M}}\phi_1\frac{\partial \phi_1}{\partial \zeta} = -\frac{\mathcal{M}}{2}\frac{\partial \phi_2}{\partial \zeta},\\
&\frac{\mathcal{M}^3}{2}\frac{\partial^3\phi_1}{\partial \zeta^3} = -\frac{\mathcal{M}^3}{2}\frac{\partial n_2}{\partial \zeta} + \frac{\mathcal{M}}{2}\frac{\partial \phi_2}{\partial \zeta} + \frac{\mathcal{M}}{2}\phi_1\frac{\partial \phi_1}{\partial \zeta},
\end{aligned} \tag{38}$$

respectively. Finally, combining Eq. (38), we find the Korteweg–de Vries equation [cf. Eq. (22) above],

$$\frac{\partial \phi_1}{\partial \eta} + B\phi_1 \frac{\partial \phi_1}{\partial \zeta} + \frac{C}{2}\frac{\partial^3 \phi_1}{\partial \zeta^3} = 0, \tag{39}$$

whose coefficients B and C may be fully expressed in terms of powers of the critical Mach number, namely,

$$B = \frac{3 - \mathcal{M}^2}{2\mathcal{M}}, \quad C = \mathcal{M}^3. \tag{40}$$

The critical Mach number may be related to physically measurable quantities.

The usual soliton solution of the Korteweg–de Vries equation [cf. Eq. (24) above],

$$\phi_1(\zeta - \beta\eta) = \mathcal{A}\,\mathrm{sech}^2\left(\frac{\zeta - \beta\eta}{\Delta/2}\right), \tag{41}$$

where β is the phase speed, when substituted in Eq. (39), shows that the wave amplitude $\mathcal{A}$ and half-width $\Delta/2$ are given by

$$\mathcal{A} = \frac{3\beta}{B}, \quad \frac{\Delta}{2} = \left(\frac{2C}{\beta}\right)^{1/2}. \tag{42}$$

Eliminating B and C between Eqs. (40) and (42), we find an algebraic, biquadratic equation for the critical Mach number,

$$\mathcal{M}^4 - 3\mathcal{M}^2 + \frac{3\beta^2\Delta^2}{4\mathcal{A}} = 0, \tag{43}$$

whose solution is given by

$$\mathcal{M} = \left[\frac{3}{2}\left(1 - \sqrt{1 - \frac{\beta^2\Delta^2}{3\mathcal{A}}}\right)\right]^{1/2}. \tag{44}$$

The minus sign in front of the square root between round brackets on the right-hand side of Eq. (44) has been chosen because such a choice correctly provides the aforementioned small, positive δ-parameter for the expansions around equilibrium in Eq. (30). In accordance with Eq. (44), we have three situations,

(i) $0 < (\beta^2\Delta^2)/(3\mathcal{A}) < 8/9 \Rightarrow 0 < \mathcal{M} < 1$,
(ii) $(\beta^2\Delta^2)/(3\mathcal{A}) = 8/9 \Rightarrow \mathcal{M} = 1$,
(iii) $8/9 < (\beta^2\Delta^2)/(3\mathcal{A}) < 1 \Rightarrow 1 < \mathcal{M} < \sqrt{3/2}$.

The lower- and upper-bounds, 0 and 1, respectively, of the combination $(\beta^2\Delta^2)/(3\mathcal{A})$ must be excluded from our analysis because all relevant functions depend on $\mathcal{M}$ and might have derivatives with respect to the potential to all orders.

A couple of clarifying comments is in order now. First, each one of the three above referred cases corresponds to a different class of reference frames. Therefore, situations (i), (ii), and (iii), determine subsonic, sonic, and supersonic, respectively, reference frames. Second, of course, the soliton speed β, amplitude $\mathcal{A}$, and half-width $\Delta/2$, all may vary widely. However, those quantities conspire in a way that the combination $(\beta^2\Delta^2)/(3\mathcal{A})$ varies between 0 and 1 only. In Sec. V, we apply our just proposed formulation to four different electron density distributions.

V. APPLICATIONS

A. Boltzmann

As a first application, consider the well-known Boltzmann distribution,

$$F_e(\Phi) = \exp\left(\frac{e\Phi}{k_B T_e}\right), \tag{45}$$

where k_B and T_e are the Boltzmann constant and electron Maxwell temperature, respectively. Equation (45) implies

$$F_e'(0) = \left(\frac{e}{k_B T_e}\right), \quad F_e''(0) = \left(\frac{e}{k_B T_e}\right)^2, \tag{46}$$

which lead to

$$\varphi = \left(\frac{k_B T_e}{e}\right), \quad \lambda_e = \left(\frac{\epsilon_0 k_B T_e}{N_0 e^2}\right)^{1/2}, \quad V_0 = \left(\frac{k_B T_e}{m_i}\right)^{1/2}, \quad \mathcal{M} = 1. \tag{47}$$

It follows from the last of Eq. (47) that the coefficients of the Korteweg–de Vries equation are given by

$$B = 1, \quad C = 1. \tag{48}$$

All the above results recover those for the usual description of soliton waves in Boltzmann plasmas by the reductive perturbation technique with respect to a sonic reference frame.[39,40] In particular, the second of Eq. (47) recovers the so-called electron Debye length. That is the meaning of the index "e" included in our notation for the length scale.

B. Kappa

We analyze now the Kappa (or Lorentzian) distribution,[41]

$$F_e(\Phi) = \left[1 + \frac{(e\Phi)/(k_B\Theta_e)}{-\kappa + 3/2}\right]^{-\kappa+1/2}, \tag{49}$$

where the so-called spectral index $\kappa > 3/2$ and the quantity Θ_e is related to the electron Maxwell temperature T_e through[42]

$$\Theta_e = \left(\frac{-\kappa}{-\kappa + 3/2}\right) T_e. \tag{50}$$

In the limit $\kappa \to \infty$, Eq. (49) recovers Eq. (45) and Eq. (50) shows that $\Theta_e \to T_e$. Since $\Theta_e \geq T_e$, the Kappa distribution is said to favor suprathermal electrons, that is, those possessing higher Maxwell temperatures. The introduction of Θ_e in the problem is necessary; otherwise, the electron thermal speed should be modified and then the speed of sound in the ionized gas would be also modified.[43,44] In that case, our proposed formulation should be modified too. Therefore, in order to avoid complications, Θ_e is simply included in the following analysis. Equation (49) implies

$$F_e'(0) = \left(\frac{-\kappa + 1/2}{-\kappa + 3/2}\right)\left(\frac{e}{k_B\Theta_e}\right),$$
$$F_e''(0) = \left(\frac{-\kappa + 1/2}{-\kappa + 3/2}\right)\left(\frac{-\kappa - 1/2}{-\kappa + 3/2}\right)\left(\frac{e}{k_B\Theta_e}\right)^2, \tag{51}$$

which lead to the potential and length scales

$$\varphi = \left(\frac{-\kappa + 3/2}{-\kappa - 1/2}\right)\left(\frac{k_B\Theta_e}{e}\right), \quad \lambda_e = \left(\frac{-\kappa + 3/2}{-\kappa - 1/2}\right)^{1/2}\left(\frac{\epsilon_0 k_B\Theta_e}{N_0 e^2}\right)^{1/2}, \tag{52}$$

respectively, as well as to the ion flow at the plasma-sheath interface and associated critical Mach number,

$$V_0 = \left(\frac{-\kappa+3/2}{-\kappa+1/2}\right)^{1/2}\left(\frac{k_B\Theta_e}{m_i}\right)^{1/2}, \quad \mathcal{M} = \left(\frac{-\kappa-1/2}{-\kappa+1/2}\right)^{1/2}, \quad (53)$$

respectively. It follows from the last of Eq. (53) that the coefficients of the Korteweg–de Vries equation are given by

$$B = \left(\frac{-\kappa+1}{-\kappa+1/2}\right)^{1/2}\left(\frac{-\kappa+1}{-\kappa-1/2}\right)^{1/2}, \quad C = \left(\frac{-\kappa-1/2}{-\kappa+1/2}\right)^{3/2}. \quad (54)$$

We arrive at the following results.

The combination of Eq. (44) and the last of Eq. (53) shows that the description of soliton waves due to suprathermal electrons by the reductive perturbation technique is possible only with respect to supersonic reference frames and ceases to be valid for $3/2 < \kappa \leq 5/2$. Such an interval for the spectral index contrasts with that previously obtained in Refs. 19 and 20 which state that the technique is not valid for $3/2 < \kappa \leq 3$. The authors have justified their result by claiming that the Taylor expansion of the suprathermal electron density distribution around equilibrium diverges for $3/2 < \kappa \leq 3$. However, they have chosen the first of Eq. (47) as the normalization for the electrostatic potential, which, as we have shown, is valid only for Boltzmann plasmas. Nonetheless, we have chosen the correct normalization for suprathermal electrons, which is given by the first of Eq. (52). In Table I, we calculate the first few low-order coefficients of the Taylor expansion for the suprathermal electron density distribution (49), normalized by the first of Eq. (52), around equilibrium for selected values of $\kappa > 3/2$. As one may easily infer, the expansion will converge for all possible values of the spectral index. This shows that it is the value of the critical Mach number, not functional analysis properties, which determines the description limit of soliton waves due to suprathermal electrons by the reductive perturbation technique.

C. Tsallis

Let us regard now the Tsallis distribution,[45]

$$F_e(\Phi) = \left[1 + (q-1)\frac{e\Phi}{k_B T_e}\right]^{[q+1]/[2(q-1)]}, \quad (55)$$

where we choose the so-called deformation index $q \leq 1$. In the limit $q \to 1$, Eq. (55) recovers Eq. (45). Equation (55) implies

$$F'_e(0) = \left(\frac{q+1}{2}\right)\left(\frac{e}{k_B T_e}\right), \quad F''_e(0) = \left(\frac{q+1}{2}\right)\left(\frac{-q+3}{2}\right)\left(\frac{e}{k_B T_e}\right)^2, \quad (56)$$

which lead to the potential and length scales,

$$\varphi = \left(\frac{2}{-q+3}\right)\left(\frac{k_B T_e}{e}\right), \quad \lambda_e = \left(\frac{2}{-q+3}\right)^{1/2}\left(\frac{\epsilon_0 k_B T_e}{N_0 e^2}\right)^{1/2}, \quad (57)$$

respectively, as well as to the ion flow at the plasma-sheath interface and associated critical Mach number,

$$V_0 = \left(\frac{2}{q+1}\right)^{1/2}\left(\frac{k_B T_e}{m_i}\right)^{1/2}, \quad \mathcal{M} = \left(\frac{-q+3}{q+1}\right)^{1/2}, \quad (58)$$

respectively. It follows from the last of Eq. (58) that the coefficients of the Korteweg–de Vries equation are given by

$$B = \left(\frac{2q}{q+1}\right)^{1/2}\left(\frac{2q}{-q+3}\right)^{1/2}, \quad C = \left(\frac{-q+3}{q+1}\right)^{3/2}. \quad (59)$$

We arrive at the following results.

The combination of Eq. (44) and the last of Eq. (58) shows that the description of soliton waves due to nonextensive electrons by the reductive perturbation technique is possible only with respect to supersonic reference frames and ceases to be valid for $q \leq 3/5$. In Table II, we calculate the first few low-order coefficients of the Taylor expansion for the nonextensive electron density distribution (55), normalized by the first of Eq. (57), around equilibrium for selected values of $q \leq 1$. As one may easily infer, the expansion will converge for all possible values of the deformation index. This shows that it is the value of the critical Mach number, not functional analysis properties, which determines the description limit of soliton waves due to nonextensive electrons by the reductive perturbation technique.

Since suprathermal and nonextensive solitons are both described by supersonic reference frames, let us assume that the parameters of those waves be measured with respect to the same coordinate system. In that case, we must require that the last of Eq. (53) be identical to the last of Eq. (58), from which it follows that the spectral and deformation indexes might be related as $q = 1 - 1/\kappa$, where κ is the spectral index introduced in Eq. (49). Such a restriction of the Tsallis to Kappa

TABLE I. Coefficients $\mathcal{O}(j)$, for order $j = 1; \cdots 6$, of the Taylor expansion for the suprathermal electron density distribution (49), normalized by the first of Eq. (52), around equilibrium for $\kappa = 2.25$; 2.50; 2.75; 3.00; 3.25. As one may easily infer, the expansion will converge for all possible values of the spectral index. This shows that it is the value of the critical Mach number, not functional analysis properties, which determines the description limit of soliton waves due to suprathermal electrons by the reductive perturbation technique.

κ-index	2.25	2.50	2.75	3.00	3.25
$\mathcal{O}(1)$	0.64	0.67	0.69	0.71	0.73
$\mathcal{O}(2)$	0.32	0.33	0.35	0.36	0.37
$\mathcal{O}(3)$	0.14	0.15	0.15	0.15	0.15
$\mathcal{O}(4)$	0.06	0.06	0.06	0.06	0.06
$\mathcal{O}(5)$	0.03	0.02	0.02	0.02	0.02
$\mathcal{O}(6)$	0.01	0.01	0.01	0.01	0.01

TABLE II. Coefficients $\mathcal{O}(j)$, for order $j = 1; \cdots 6$, of the Taylor expansion for the nonextensive electron density distribution (55), normalized by the first of Eq. (57), around equilibrium for $q = 0.40$; 0.50; 0.60; 0.70; 0.80. As one may easily infer, the expansion will converge for all possible values of the deformation index. This shows that it is the value of the critical Mach number, not functional analysis properties, which determines the description limit of soliton waves due to nonextensive electrons by the reductive perturbation technique.

q-index	0.40	0.50	0.60	0.70	0.80
$\mathcal{O}(1)$	0.54	0.60	0.67	0.74	0.82
$\mathcal{O}(2)$	0.27	0.30	0.33	0.37	0.41
$\mathcal{O}(3)$	0.13	0.14	0.15	0.16	0.16
$\mathcal{O}(4)$	0.06	0.06	0.06	0.06	0.06
$\mathcal{O}(5)$	0.03	0.03	0.02	0.02	0.02
$\mathcal{O}(6)$	0.01	0.01	0.01	0.01	0.00

distributions has been previously obtained on very fundamental grounds.[46,47] This strongly suggests that other possible consequences of our proposed formulation to the general soliton theory deserve further investigation.

D. Fermi

As a final application, let us examine a Fermi gas of $\mathcal{N}_e$ electrons with mass m_e, which occupies a volume $\mathcal{V}$ at the Fermi temperature T_F. The Fermi energy of the gas may be expressed through[48]

$$k_B T_F = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2}{\mathcal{V}/\mathcal{N}_e}\right)^{2/3}, \tag{60}$$

where $\hbar = h/(2\pi)$ is the normalized Planck constant, with h denoting the Planck constant. To be specific, consider the Thomas-Fermi distribution,[49,50]

$$F_e(\Phi) = \left(1 + \frac{e\Phi}{k_B T_F}\right)^{3/2}. \tag{61}$$

Equation (61) implies

$$F_e'(0) = \frac{3}{2}\left(\frac{e}{k_B T_F}\right), \quad F_e''(0) = \frac{3}{4}\left(\frac{e}{k_B T_F}\right)^2, \tag{62}$$

which lead to

$$\begin{aligned} \varphi &= \left(\frac{2k_B T_F}{e}\right), \quad \lambda_e = \left(\frac{2\epsilon_0 k_B T_F}{N_0 e^2}\right)^{1/2}, \\ V_0 &= \left(\frac{2k_B T_F}{3m_i}\right)^{1/2}, \quad \mathcal{M} = \left(\frac{1}{3}\right)^{1/2}. \end{aligned} \tag{63}$$

It follows from the last of Eq. (63) that the coefficients of the Korteweg–de Vries equation are given by

$$B = 4\left(\frac{1}{3}\right)^{1/2}, \quad C = \frac{1}{3}\left(\frac{1}{3}\right)^{1/2}. \tag{64}$$

We arrive at the following results.

It has been shown that the Mach number may be fully expressed in terms of a perturbation α ($|\alpha| \ll 1$) on the amplitude of a soliton that propagates in a Fermi gas,[51]

$$\mathcal{M} = 4(1+\alpha)^{1/3} - 3, \tag{65}$$

where $\alpha < 0$ describes a subsonic (or dark) soliton, $\alpha = 0$ describes a sonic soliton, and $\alpha > 0$ describes a supersonic (or bright) soliton. Given the last of Eq. (63), we expect the amplitude perturbation to be negative in our problem. To check whether that is true, we substitute the last of Eq. (63) in Eq. (65) to actually find $\alpha \approx -0.28$. Such a result offers independent confirmation that our proposed formulation succeeds in describing the soliton propagation in a Fermi gas with respect to a subsonic frame. Moreover, this reveals that our provided theory may be applicable even in semiclassical contexts, which are appropriate, for instance, to describe phenomena in dense plasmas.

VI. CONCLUSION

We have provided a formulation that describes the propagation of solitons in a nondissipative, nonmagnetic plasma, which does not depend on the particular electron density distribution considered. The Poisson equation in the plasma sheath has been expressed in terms of the Mach number for ions entering the sheath from the plasma and of a natural scale for the electrostatic potential.

We have found a class of reference frames with respect to which certain functions become stationary after arbitrary small variations of the Mach number and potential scale, that is, by determining the critical values of those quantities based on a variational method. It has been shown that the critical Mach number defines the limits for the applicability of the reductive perturbation technique to a given electron density distribution.

We have applied our proposed formulation to four different electron distributions. First, the usual results for the Boltzmann distribution have been recovered.

Then, based on our provided potential scale, we have shown that the Taylor expansion of the suprathermal electron distribution around equilibrium converges for all possible values of the spectral κ-index. In addition, owing to the admissible range for the critical Mach number, it has been found that the reductive perturbation technique ceases to be valid for $3/2 < \kappa \leq 5/2$.

In the sequel, we have shown that the technique is not valid for the deformation q-index of nonextensive electrons when $q \leq 3/5$. Furthermore, by assuming that the suprathermal and nonextensive solitons are both described with respect to the same critical reference frame, a relation between κ and q, which has been previously obtained on very fundamental grounds, has been recovered.

Finally, we have found that our model may be applied even to a Fermi gas. Moreover, independent confirmation of that has been offered.

To summarize, we stress that our proposed analytical formulation, namely, the critical scaling of the electrostatic potential, and the modified reductive perturbation technique, has two important consequences. First, it provides a unifying approach to the description of soliton dynamics, thereby avoiding unnecessary efforts and contradictory results in this topic, as may be found in the literature. Second, it shall serve as a starting point to further investigations in nonlinear structures.

The first, natural extensions of our theory should include magnetic and dissipative effects. Application of the reductive perturbation technique has shown that dense electron-positron plasmas may support Korteweg–de Vries solitons associated with magnetosonic modes.[52] In particular, it has been found that no soliton solution exists at certain critical angles between the species flow and magnetic field.

Strongly coupled, dusty plasmas have also been shown to support Korteweg–de Vries solitons in the presence of a magnetic field.[53] In that case, certain forces have been identified as the cause of the soliton decay.

The reductive perturbation technique applied to dense astrophysical environments, such as neutron stars and pulsar magnetospheres, shows the emergence of both oscillatory and monotonic shock waves as solutions of a Korteweg–de Vries-Burgers-type equation.[54] In particular, different plasma parameters have been found to perform a relevant role in the transition between the oscillatory and monotonic regimes.

Self-sustained pulses have been found as solutions of modified Korteweg–de Vries (or Burgers) equations in plasmas with finite thermal conductivity.[55] In that case, it has been shown that nonadiabaticity affects the establishment of stationary wave configurations. All the above referred issues shall be investigated in further communications.

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The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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