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Dynamics of the Kicked Logistic Map

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Abstract—The modulation of the logistic map by a sequence of periodic kicks brings up a three-parameter kicked logistic map (klm) with new distinct dynamic features. Thus, its parameter space structure exhibits highly interleaved sets with different attractors, and complex basins of attraction are created. Additional roots to chaos and abrupt attractor changes are identified in the parameter space. The observed intermittency route to chaos is distinct to those typical of spatially discontinuous unidimensional maps, with a characteristic power-law dependence of the average laminar length on the control parameters. This behaviour is verified for both the internal and the transfer crisis-induced intermittency.

1. INTRODUCTION

Periodic and chaotic non-linear dissipative systems can show peculiar response behaviour to applied forces of various types [1–4]. Thus, for relevant control parameter sets of these systems, striking changes in their dynamics are induced even for weak perturbations [5, 6]. Some of these alterations identified in the modified bifurcation diagrams are: control of chaos, new roads to chaos, abrupt attractor changes and new types of crisis. These features have been found in numerical experiments performed with several dynamical systems modelled either with differential or difference equations modulated by random as well as periodic forcing. Furthermore, these properties have also been observed experimentally in several dissipative dynamical systems in many scientific disciplines [2–4].

In dynamics, problems described by non-linear differential equations are often reduced to discrete maps by considering their Poincaré sections. In particular, unidimensional maps have been the subject of increasing interest, both due to their intrinsic mathematical richness and to the large number of dynamical systems that experimentally display transitions into chaos through the universal bifurcation scenario [2, 3, 7]. The most known and extensively studied of these maps is the logistic map.

In the last years, studies of the effect of forcing on systems simulated by the logistic map have been reported [8–11]. Thus, a new type of crisis, associated with the hysteresis followed from the coexistence of two attractors, was found in this map with a periodic modulation [11]. The effect of additive and multiplicative noise on the first bifurcations of the logistic model was analytically and numerically studied [10]. Periodic entrainment of chaotic trajectories was discussed in [9]. For additive periodic forcing two non-complementary attractors were found [8]. However, since most of these interesting results have been obtained by changing single parameters, this subject is still far from being fully explored. In particular, it seems worthwhile to look for new dynamic properties regarding relevant parameter sets for which this system behaves similarly concerning one or several properties of interest, such as, for example, periodicity of attractors and basins of attraction. Thus, new fundamental dynamic properties may be recognized and interpreted performing the analysis in the parameter space, as it has been recently done for other systems [12, 13].

The aim of the present paper is to numerically study the main topological changes of the logistic map attractors caused by a sequence of constant kicks. Examples of structures formed by the domain of the chaotic and periodic attractors in the space of the control parameters (the kick amplitude and the control parameter of the logistic map) are presented for given kick periods. These pictures reveal highly interleaved regions corresponding to parameter sets for which finite attractors exist. Another remarkable alteration induced by the kicks appears in the intermittency route to chaos (characterized by dynamical intermittent changes between laminar periods and chaotic bursts). For the kicked logistic map (klm), i.e., the perturbed map considered in this paper, besides the interior crises [14], transfer crises [11] have also been observed. For this transfer crisis, a power dependence of the average length of the laminar phase on the control parameters is numerically obtained, in contrast to the spatially discontinuous maps [17–19]. Other dynamic properties of this kicked logistic map, such as suppression of chaos, periodic entrainment, routes to chaos and bifurcation diagrams, were considered elsewhere [15, 16].

Section 2 presents attractor regions in the parameter space of the kicked logistic map for different kick periods. Section 3 presents examples of interior and transfer crises and describes numerical experiments that determine the scaling of the average laminar length with control parameters. Finally, discussion is given in Section 4.

2. STRUCTURE OF THE PARAMETER SPACE

In this paper we consider the following kicked logistic map, which corresponds to the logistic map perturbed by a sequence of kicks with a constant amplitude q and a period t:

$$X_{n+1} = bX_n(1 - X_n) + q\delta_{n,t}, \tag{1}$$

where $\delta_{n,t} = 1$ if n/t is an integer and 0 if not. This system has finite attractors corresponding to periodic or chaotic oscillations in the interval

$$0 < X_n < 1. \tag{2}$$

To investigate numerically the control of the logistic map oscillations through a sequence of constant kicks, the quantification of chaos and order is obtained by computing the Lyapunov exponent, λ, for the kicked logistic map trajectories. This is computed from the following expression obtained from equation (1):

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln |b(1 - 2X_n)| \tag{3}$$

where X_n is obtained from equation (1).

The order (i.e. predictability) is indicated by $\lambda < 0$, and the chaos (i.e., sensitive dependence on initial conditions) is indicated by $\lambda > 0$.

In this section we consider sets of parameters, b, q, and t, for which the klm trajectories are periodic or chaotic. In order to observe the sensitive dependence of the attractor on the control parameter, much attention was paid to obtain high-precision figures. Thus, before applying equation (3) to compute λ, 1000 iterations are performed in order to allow transients to die away; after that satisfactory convergence of λ is achieved by setting $N = 3000$. However, for parameters near critical values, as those corresponding to abrupt attractor changes or bifurcations, longer transients were considered ($N \geqslant 100\,000$).

Figures 1 and 2 show typical attractor regions in the $b \times q$ space for odd ($t = 5$) and even ($t = 2$) kick periods, respectively. In these figures black and white pixels represent, respectively, parameters for which the klm trajectories are chaotic and periodic, while gray

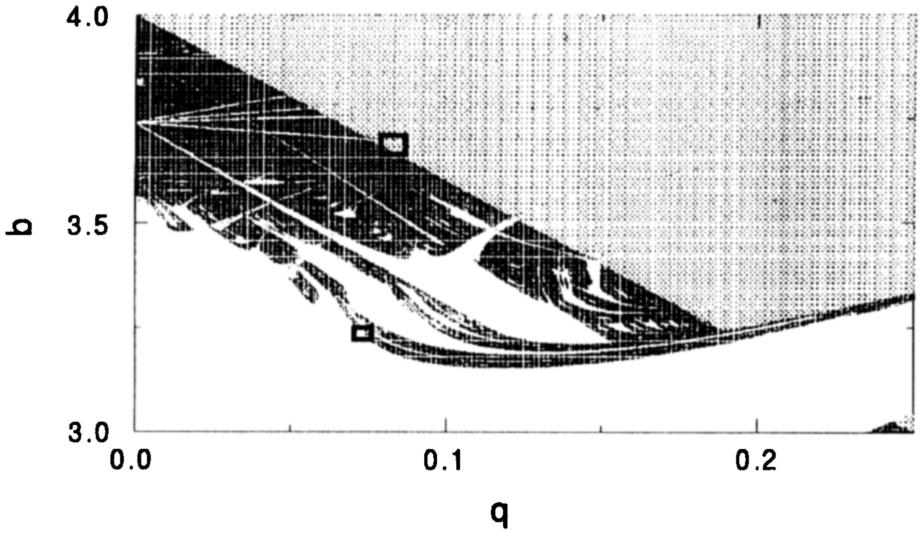


Fig. 1. Attractor regions in the parameter space for kick period $t = 5$ with black and white pixels representing chaotic and non chaotic attractors. Gray pixels represent points without limited attractor.

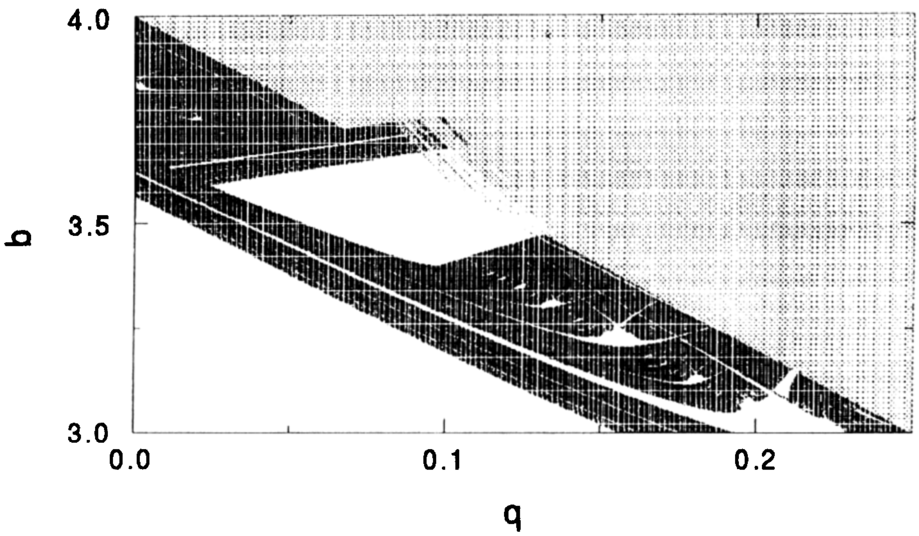


Fig. 2. Attractor regions in the parameter space for kick period $t = 2$ with black and white pixels representing chaotic and non chaotic attractors. Gray pixels represent points without limited attractor.

pixels represent points without any limited attractor (i.e., $X_n \rightarrow \infty$, as $n \rightarrow \infty$). The computed regions are complex and highly interleaved. The figures obtained for other periods t present patterns similar to those two shown in Figs 1 and 2. Regarding the observed similarities in their main characteristic shapes and structures, the computed figures can be classified in two groups, one of them with odd t and the other with even t. These similarities persist even for high t values, for which the chaotic regions are less dense.

In these figures, finite attractors appear for b, q values with pixels under the line described by the equation

$$b/4 + q = 1. \tag{4}$$

On the other hand, for parameters represented by pixels in the area above this line, most of these displayed pixels indicate the existence of infinite attractors. However, this region also contains finite attractors, dependent on the initial value X_0, indicated by rarefied clusters of black and white points.

The dependence of the parameter space structures on the initial value X_0 is illustrated in Figs 3 and 4 that show magnifications of two areas indicated in Fig. 1, one above (A) and the other under (B) the line defined by equation (4). These figures were computed for the same parameters but for two different initial conditions. Figures 3(A) and 4(B) show significant differences, while Figs 3(B) and 4(B) (for the chosen X_0) are exactly the same. For the former case, by fixing b or q, one would get broken bifurcation diagrams [16]. However, besides the initial condition, these conclusions depend also on the chosen control

(a)

(b)

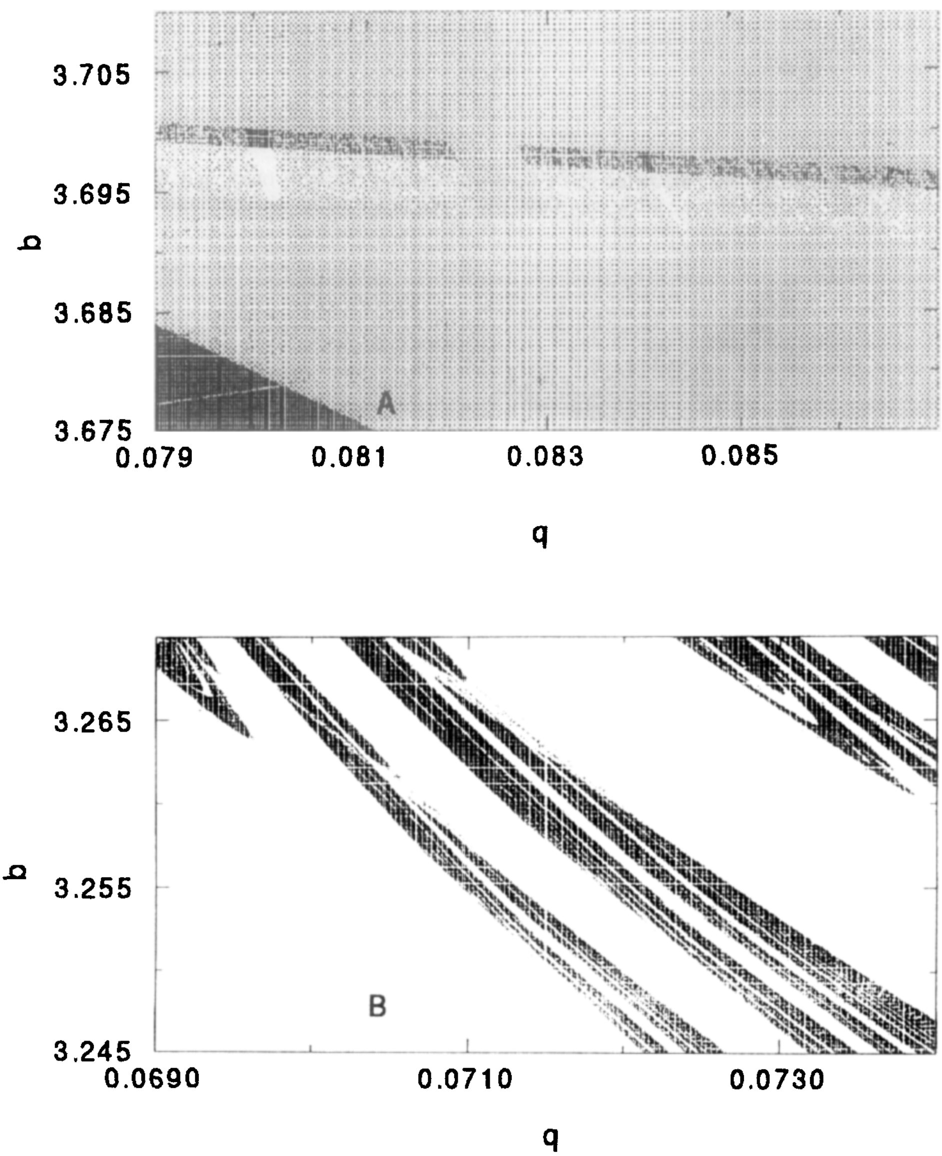
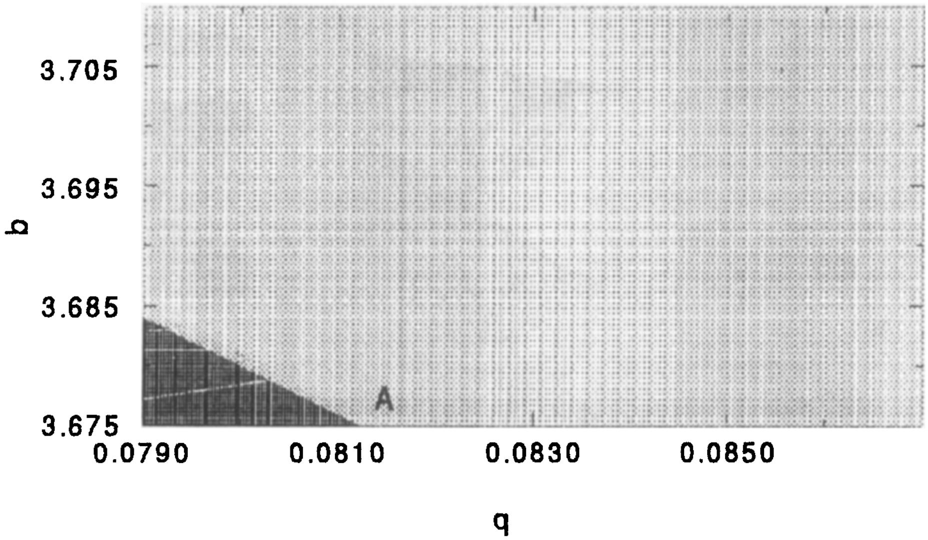


Fig. 3. Magnifications of the two regions indicated by the up (A) and down (B) squares in Fig. 1, for $X_0 = 0.2$ and $t = 5$.

(a)



(b)

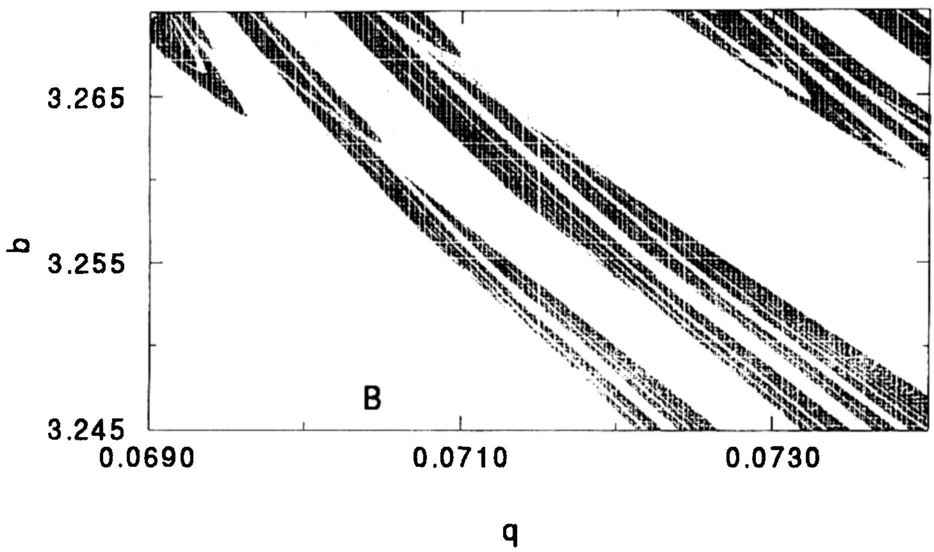


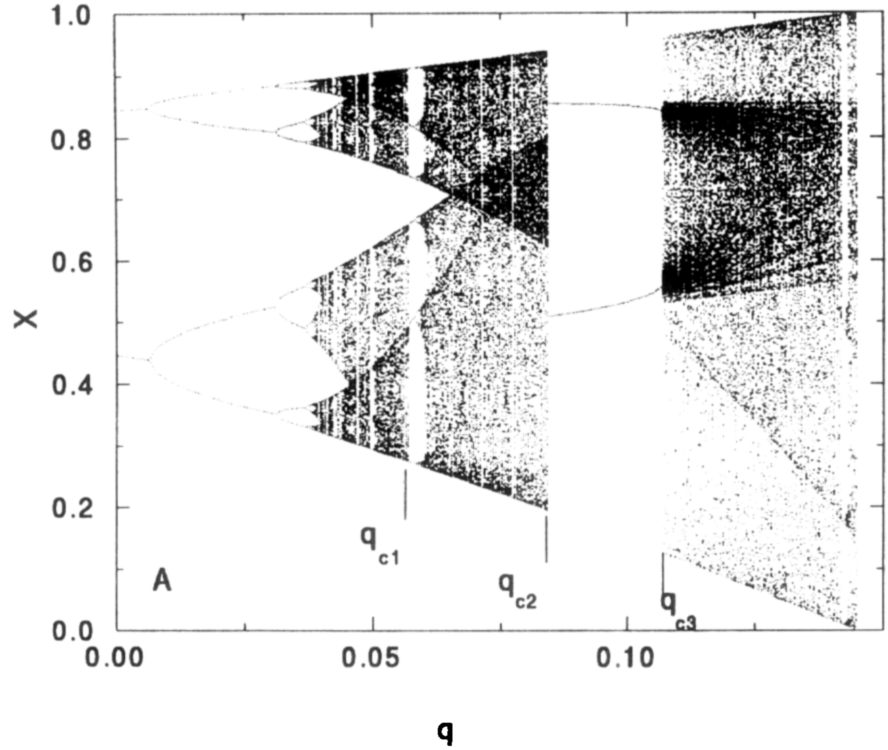
Fig. 4. Magnifications of the two regions indicated by the up (A) and down (B) squares in Fig. 1, for $X_0 = 0.3$ and $t = 5$.

parameters. Moreover, for any b, q values with pixels in the area above the line described by equation (4), there is no finite attractor if X_n, during the transient, assumes any value in the following interval:

$$1/2[1 + \sqrt{1 + b/4(q - 1)}] > X_n > 1/2[1 - \sqrt{1 + b/4(q - 1)}] \quad (5)$$

It is seen from Fig. 5(B) that there is a period-two stable attractor for $q < q_{c2}$. In fact, there are two coexisting attractors for these values of t, b, and q with different basins of attraction $\{X_0\}$. That is, the chaotic band and the stable periodic orbit coexist together for $q < q_{c3}$. Thus, whether the chaotic band or the periodic orbit is realized directly depends on the choice of an intial value X_0 in the proper basin of attraction. This sudden transfer

(a)



(b)

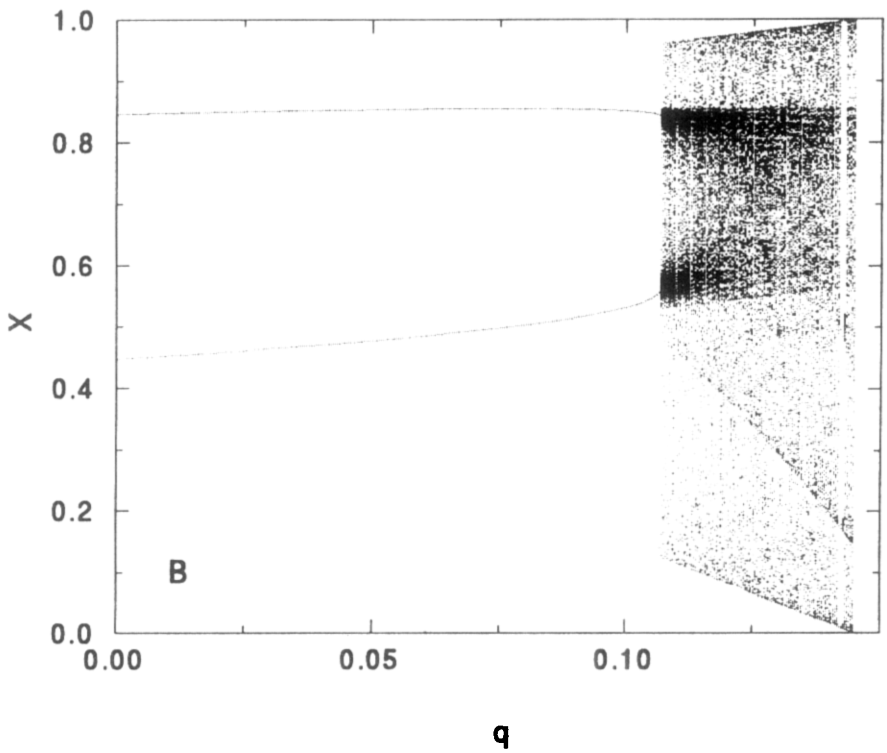


Fig. 5. (A) Bifurcation diagram with kick period $t = 2$, $b = 3.42$ and $X_0 = 0.2$. Three critical perturbation amplitude values are indicated. (B) Bifurcation diagram for the same control parameters and another initial condition $X_0 = 0.45$.

of orbit has been called transfer crisis in [11]. The main difference between the interior and transfer crisis is whether there is a stable fixed point or not beyond the repeller. That is, the transfer crisis is associated with the coexistence of peroidic and chaotic attractors.

Figure 6(A) shows, for $t = 2$ and $q = 0.12$, another sudden transition at $b_c = 3.208093$. In this interior crisis there is a transition from a chaotic to a stable period-six attractor that can be already recognized in the laminar phase (Fig. 6(B)). In addition, Fig. 6(B) shows a chaotic orbit with laminar phases similar to unstable orbits, with period two and four, indicated in the boxes. For $b > b_c$, the emerging period-six orbit exhibits a sequence of bifurcations before the system enters into chaos. However, this sequence is different from the Feigenbaum pitchfork bifurcation scheme. In contrast to this scenario, intersecting periodic orbits are additionally observed in the corresponding region of the analysed bifurcation diagram. This kind of crossover has also been observed for spatially discontinuous maps [22] and coupled maps [21].

Thus, this condition is satisfied for all points in the parameter space corresponding to finite attractors shown in the previous figures.

Besides the transitions into chaos via the Feigenbaum scenario, analyses of bifurcation diagrams of spatially discontinuous unidimensional dissipative maps showed other roads to chaos [17–19]. For the investigated klm map, although the Feigenbaum scenario persists in considerable parameter ranges, it can be strongly modified depending on the kick period and amplitude variations. So, as one can notice from Figs 1 to 4, new roads can be identified by analysing, in the parameter space, the modification of the attractor periods and the entrances into chaos associated with specific control parameter variations.

Afterwards, to investigate the evolution of the attractor by increasing only one parameter, magnifications of bifurcation diagrams in proper parameter sets were also examined.

The observed attractor period, p, is always a multiple of the perturbation period, t. For increasing control parameter, b or q, these attractors exhibit a sequence of peroidic orbits with periods $2^m p$ ($m = 0, 1, 2, \ldots$), which appears through pitchfork bifurcations of the orbits with period $2^{(m-1)}p$. Thereafter, for further increasing control parameter the attractor undergoes a cascade to chaos with the same Feigenbaum constants as those of the non-perturbed logistic map [2].

As for discontinuous unidimensional maps [17–19], inverse cascades have also been observed increasing q in the klm bifurcation diagrams. Contrasting with some periodic orbits of the former maps, for the observed klm orbits the periods p' decrease only geometrically, i.e., $p' = 2^{(m-k)}Nt$, where k represents the number of times that the system suffered inverse cascades. Moreover, there are no windows with discontinuous p-bifurcations whose periods increase arithmetically, as for those discontinuous maps [17]. After all, inverse cascades were only observed, for fixed t and b, in the bifurcation diagrams with q as the growing parameter.

Abrupt disappearance of finite attractors and abrupt entrances into chaos were also observed in the growing q bifurcation diagrams. Thus, in these investigated diagrams, the mentioned period p attractors can proceed away from chaos, besides following a sequence of period-doubling bifurcations. In this last case, the attractor period p can also be odd.

3. CRISIS-INDUCED INTERMITTENCY

For a low-dimensional dynamic system intermittency is the occurrence of the alternating bursts of almost regular (so-called laminar) periods and chaotic bursts in its long-time behaviour, for a small range of a control parameter. For these systems, different types of intermittency were studied by Pomeau and Maneville [20]. Concomitantly, crisis-induced

(a)
(b)

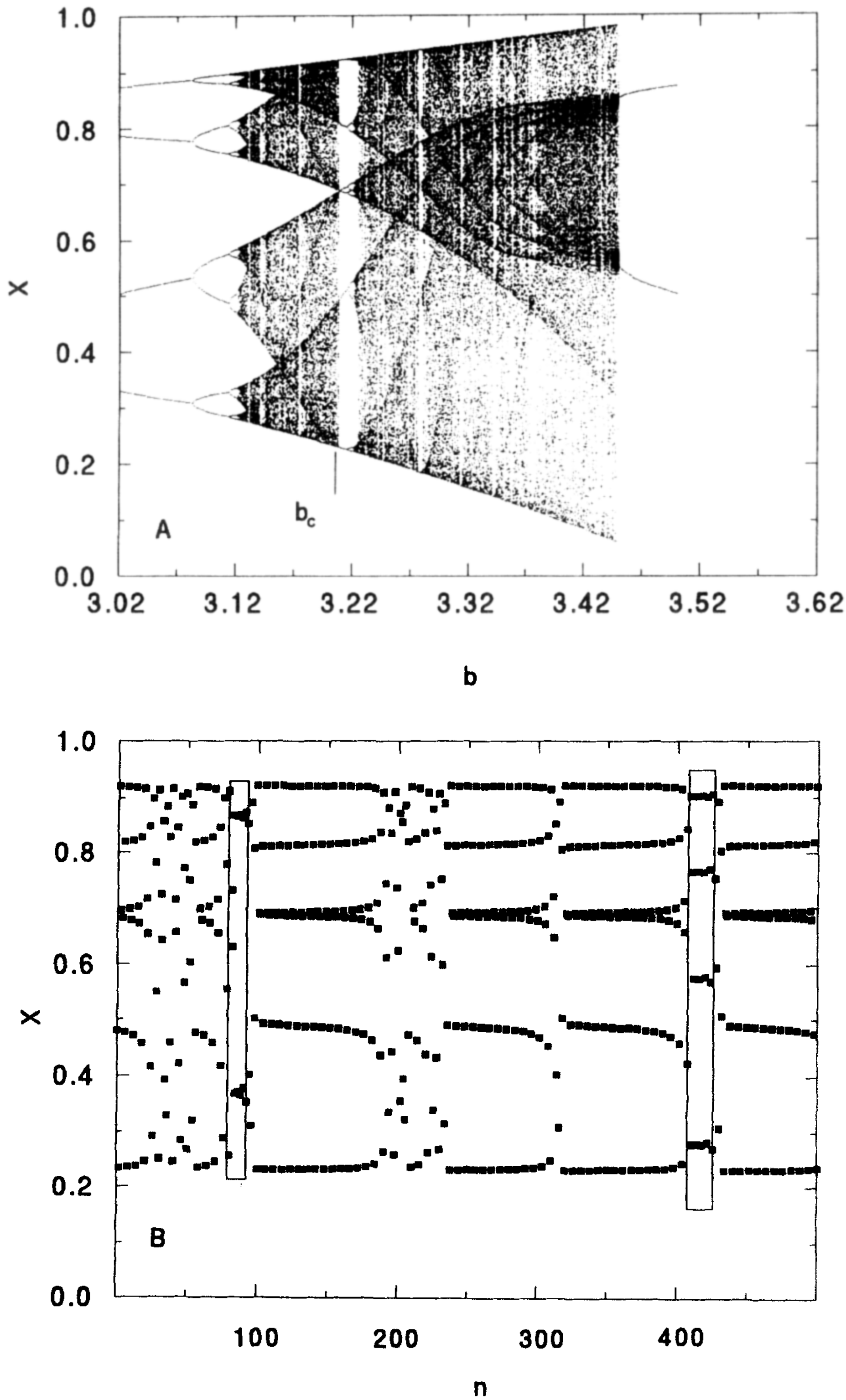


Fig. 6. (A) Bifurcation diagram with kick period $t = 2$ and $q = 0.12$. $b_c = 3.208093$. (B) Temporal evolution of the kicked logistic map for $t = 2$, $q = 0.12$ and $b = 3.208$. The two boxes show unstable period-two and period-four oscillations.

intermittency [14] has received considerable attention as an example of type-I intermittency [23] and one of the typical routes to chaos.

In the logistic map an interior crisis occurs whenever a chaotic attractor collides with an unstable fixed point in a tangent bifurcation [14]. In the bifurcation diagram, this crisis is characterized by a transition from narrow bands to a wide band and vice versa at critical values, b_c, of the control parameter. Moreover, this attractor-merging crisis induces characteristic temporal behaviour typical of type-I intermittency transitions. In this case, for the varying control parameter near this critical value, the chaotic motion is characterized by intervals of approximately periodic behaviour, the so-called laminar phase, similar to the resultant periodic attractor. This motion is interrupted by finite-duration chaotic bursts occurring at irregular times, but one can define a time, $\langle l \rangle$, for the laminar phase between the bursts. As the control parameter approaches its critical value (in the chaotic region), this mean time approaches infinity, and the attractor orbit thus becomes always laminar so that the motion is periodic. On the other hand, as the control parameter proceeds away from its critical value, the bursts become so frequent that the laminar phase can no longer be distinguished [24].

This temporal behaviour is not observed in asymmetric logistic maps, for which the periods doubling bifurcations appear discontinuously like tangent bifurcations but do not present intermittency [17]. Contrary to the spatial asymmetry introduced in these maps, the modulation of the logistic map by additive periodic forcing does not remove the interior crises yet. Thus, for a fixed perturbing period t in the klm, there are sudden transitions (similar to those observed in the logistic map) from a periodic to a chaotic attractor at the critical parameter, b_c or q_c, when only one of the two control parameters, b or q, is varied. One example of this transition can be seen in Fig. 5(A) at $q = q_{c1}$.

For the klm orbits, $\{X\}$, besides the previous mentioned crisis, there are other types of interior crises that induce intermittency. Thus, bifurcation diagrams for this map are plotted in Fig. 5 with the initial conditions $X_0 = 0.2$ (A) and $X_0 = 0.45$ (B), $t = 2$, and $b = 3.42$. There are three discontinuous changes of orbits, indicated in Fig. 5(A), at $q_{c1} \cong 0.0568$, $q_{c2} \cong 0.0843$, and $q_{c3} \cong 0.1068$. As it can be seen in this figure, the X values computed in the laminar phase, for $q < q_{c2}$, are not equivalent to those computed for the merging period-two attractor. That is, the chaotic orbit, during the laminar phase, does not stay in the same region of the periodic attractor. Therefore, contrasting to the interior crisis observed at $q_{c1} \cong 0.0568$ (similar to that observed in the logistic map), no chaotic attractor colliding with an unstable fixed point (in a tangent bifurcation) can be associated with this new type of crisis. Additionally, similar transitions are seen in other windows. The other transition at $q = q_{c3}$ occurs between a period-two and chaotic attractors. In this case, after the transition, an orbit typically spends in the chaotic region long stretches of time moving chaotically in the region of the old period-two attractors.

The type-I intermittency transition to chaos displays, near critical parameters, a characteristic power-law dependence of γ on the control parameter [23]. Thus, considering a control parameter, g, the scaling of the average time of the laminar phases is given by

$$\langle l \rangle \sim (g - g_c)^{-\gamma} \tag{6}$$

for g just past g_c, where γ is the critical exponent of the crisis and g represents b or q.

The average laminar length was computed for several klm orbits, merging to a period-six attractor, for some regions of the space parameter showed in Fig. 2 ($t = 2$). Thus, by iterating the klm for different fixed values of b or q, one obtains Fig. 7, which shows the computed logarithm of γ versus the logarithm of the distance $\varepsilon = |g - g_c|$. Since five orders of magnitude of the control parameters were analysed, the power law of equation (6) is well verified for the investigated intermittency. Thus, the results show no logarithmic

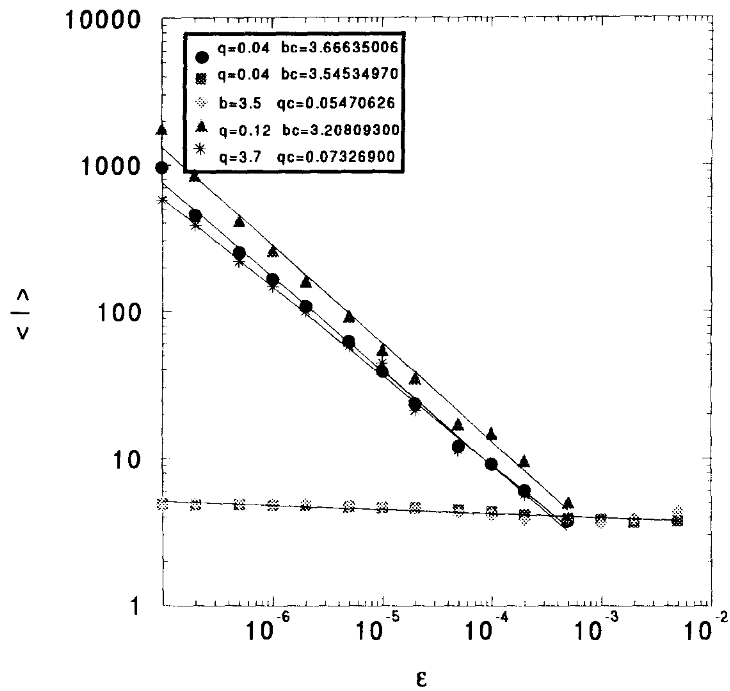


Fig. 7. The average laminar length $\langle l \rangle$ for different values of the parameters q and b, indicated in the box, and $t = 2$.

dependence of γ on any control parameter, as it was observed for spatially discontinuous maps [17]. Moreover, in this figure, the squares and diamonds represent intermittency transitions induced by transfer crises as shown in Fig. 5(A). For these transitions, the average laminar length does not change much with the parameter q. In these cases, the average laminar lengths are lower than those computed for the other transitions to period-six attractors induced by interior crisis.

Thus, the results show no logarithmic dependence of γ on any control parameter, as it was observed for spatially discontinuous maps [17].

4. CONCLUSIONS

The modulation of the logistic map by a sequence of periodic kicks introduces new distinct dynamic features described by the kicked logistic map (klm). In this work, some of these characteristics were investigated by determining, in the three-dimensional parameter space, sets of parameters, b, q, and t, for which the klm finite attractors exist and behave similarly with respect to their periodicity. These sets constitute complex and highly interleaved structures in the parameter space, some of them presented in this paper. Nevertheless, regarding the observed similarities in their main characteristic shapes and structures, the computed figures can be classified in two groups, one of them with odd kick period t and the other with even t. Furthermore, complex basins of attraction are created in the parameter range for which finite attractors may not exist.

Although the Feigenbaum scenario persists in considerable parameter ranges, it can be strongly modified depending on the kick period and amplitude variations. So, as one can notice from the figures presented in this work, new roads can be identified by analysing, in the parameter space, the modification of the attractor periods and the entrances into chaos associated with specific control parameter variations.

As for discontinuous unidimensional maps [17–19], inverse cascades have also been observed increasing q in the klm bifurcation diagrams. Contrasting with some periodic orbits of the former maps, with discontinuous p-bifurcations whose periods increase arithmetically [17], for the observed klm orbits the periods p decrease only geometrically. After all, inverse cascades were only observed, for fixed t and b, in the bifurcation diagrams with q as the growing parameter.

Abrupt disappearance of finite attractors and abrupt entrances into chaos [25] were also observed in the growing q bifurcation diagrams. Thus, in these investigated diagrams, the mentioned period p attractors can proceed away from chaos, besides following a sequence of period-doubling bifurcations. Contrary to the spatial asymmetry introduced in these maps, the modulation of the logistic map by additive periodic forcing does not remove these interior crises yet.

For the klm orbits, besides the mentioned interior crisis, there are transfer crises that also induce intermittency. Contrasting with the interior crisis (similar to that observed in the logistic map), no chaotic attractor colliding with an unstable fixed point (in a tangent bifurcation) can be associated with this crisis. The main difference between the interior and transfer crisis is whether there is a stable fixed point or not beyond the repeller. That is, the transfer crisis is associated with the coexistence of periodic and chaotic attractors. Thus, whether the chaotic band or the periodic orbit is realized directly depends on the choice of an initial value X_0 in the proper basin of attraction. This dependence on X_0 explains the hysteresis observed in the numerical experiences.

The dependence of the average length of the laminar phase with the control parameters is similar to that observed to the logistic map, but distinct to those typical of spatially discontinuous unidimensional maps. This was numerically observed for the intermittency induced by internal and transfer crisis.

The results reported in this paper may be useful to interpret results which have been obtained in experiences of control through impulsive perturbations [2–4, 26].

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