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Easy-to-implement method to target nonlinear systems

Murilo S. Baptista[a] and Iberê L. Caldas
Institute of Physics, University of São Paulo, C. P. 66318, CEP 05315-970 São Paulo, S.P., Brazil

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In this work we present a method to rapidly direct a chaotic system, to an aimed state or target, through a sequence of control perturbations, with few different amplitudes chosen according to the allowed control-parameter changes. We applied this procedure to the one-dimensional Logistic map, to the two-dimensional Hénon map, and to the Double Scroll circuit described by a three-dimensional system of differential equations. Furthermore, for the Logistic map, we show numerically that the resulting trajectory (from the starting point to the target) goes along a stable manifold of the target. Moreover, using the Hénon map, we create and stabilize unstable periodic orbits, and also verify the procedure robustness in the presence of noise. We apply our method to the Double Scroll circuit, without using any low-dimensional mapping to represent its dynamics, an improvement with respect to previous targeting methods only applied for experimental systems that are mapping-modeled. © *1998 American Institute of Physics.* [S1054-1500(98)02101-6]

In general, the sensitivity of chaotic systems to small perturbations can be used both to stabilize regular dynamic behaviors and to rapidly direct chaotic orbits to an aimed state or target. Here we present a method to target nonlinear systems *via* a sequence of control perturbations, with a few different amplitudes chosen from the set of allowed values. Consequently, a large amount of memory is not needed to determine this sequence. As examples, we apply this method to target systems represented by maps, differential equations, or a given set of data. This procedure is also shown to be robust in the presence of noise.

I. INTRODUCTION

Chaotic evolution of dynamic variables always reaches any finite range of values (within allowed variations) necessary for particular interests.[1] However, in general, the time intervals for that are too long for practical applications or investigations.[2] In order to decrease this waiting time, perturbations are applied to direct systems to desired states.[3] Particularly, chaotic systems present the advantage of being sensitive to any arbitrary (even small) perturbation on their control parameters, which may introduce enormous alterations in the dynamical variables evolution.[4] Therefore, such property of chaotic systems can be used to target system variables to required values suitable to desired applications. As practical applications we mention nave targeting,[5,6] submarine sensors,[7] and communication.[8]

In Ref. 9 the authors applied small perturbations on a parameter, without modifying the original dynamical system, to control an unstable periodic orbit. Since then, many other similar methods to control chaos have been proposed.[10–14] Some of these methods control chaos by applying a resonant perturbation that originates a stable phase locked trajectory.[15–17] However, in some cases, the resonant perturbations to eliminate chaos are large and, therefore, they modify the original dynamics.

In general, to apply some methods of chaos control the trajectory of the system to be controlled needs to be on some desired point, namely, in the vicinity of the unstable periodic orbit one wants to control. However, it may be necessary to wait a long time for a chaotic system to reach this point. So, targeting methods have been proposed to rapidly direct a chaotic trajectory to some specific location.

The idea of targeting a chaotic trajectory is owned to Shinbrot *et al.*,[3] who used the Hénon map to demonstrate their method. This method considers one determined initial perturbation, which will direct a trajectory from a starting point to a desired target. If the dynamical equations are unknown, the method can only be used if the system can be accurately modeled.[7,18]

For the Lorenz system, a three-dimensional flow with one positive Lyapunov coefficient, Shinbrot *et al.*[19] applied the method to direct flows to stationary states. However, this method is not useful if the system is high dimensional, that is, the system has more than one positive Lyapunov exponent.[20,21] So, in Ref. 22, a method is presented that can be applied to high-dimensional systems with known equations, considering one determined perturbation for each positive Lyapunov coefficient.

These previously mentioned methods do not focus the question of targeting most efficiently a chaotic system. On the other hand, in Refs. 23, 24 the authors use optimal control theory to target the Hénon map, in the fastest possible way, by applying definite perturbations with non specified amplitudes.

However, the methods of Ref. 22 and Refs. 23 and 24 require the system parameter to be changed by a very precise value, not achievable in some real systems.

So, we propose a method to target trajectories using only a few different perturbation amplitudes. Furthermore, the method we present does not need to use prescribed and high precise amplitude perturbations, but rather a sequence of per-

[a]Presently at the Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742.

turbations with few different amplitudes. And to apply numerically this idea only a simple computational programming is required, which makes this method easy to implement.

In this work, to illustrate our target method, we use three dynamical systems, the Logistic map, the Hénon map, and the Double Scroll circuit, chosen because they present common phenomena also observed in many other systems. So, the successful targeting of these three systems indicates that the method can be straightforwardly applied to other systems.

The Logistic map is a one-dimensional equation, proposed to study a biological population growth.[25,26] Despite the fact that the Logistic map is too simplistic to capture the complexity of real systems, it is a good basis for investigating such systems.[27,28].

The Hénon map is a two-dimensional equation proposed in Ref. 29 to study conservative dynamical systems such as the changing orbits of asteroids or satellites. Its dissipative form, as used in this work, allows the appearance of a strange attractor recently proved to be in fact chaotic.[30] Also, this mapping can be seen as a model for a CO_2 laser with modulated losses[31] and other systems. Furthermore, it is much used as a prototype dissipative map for numerical experiments.[32]

The last system studied here is the Double Scroll circuit,[33] an electronic circuit with a piecewise nonlinear resistor that has three energetic elements, two capacitors, and one inductor. The presence of chaos in this circuit has been observed experimentally, verified by computer simulations, and also proven mathematically. In fact, this circuit is the only real system to be proved to be chaotic,[34] besides the Hénon attractor. In addition, this circuit has been used for communicating with chaos[35] and to compose music.[36]

In Sec. II we present a general idea of our method applied to maps.

In Sec. III, using the Logistic map, we explain (using the manifold theory) how the perturbation targets this chaotic system.

In Sec. IV we use our targeting method to calculate a set of perturbations to direct the system trajectory from a point outside the nonperturbed Hénon attractor to this same point. In this case, the obtained trajectory is an unstable periodic orbit that can be stabilized. We also show the procedure robustness when noise is introduced into the system.

In Sec. V we show how to target a three-dimensional flow, namely the Double Scroll circuit,[33] without the necessity of modeling the system or using the previous knowledge of the dynamical equations. Although in this paper we only present numerical results, the numerical targeting of this circuit is done as we were dealing with a real experiment. We also use our method to targeting before applying the OGY method of control[9] in the Double Scroll system.

II. TARGETING MAPS

Consider a one-parameter map, represented by the equation $Z_{n+1}=F(b,\ Z_n)$, whose parameter b can be changed by $\pm\delta$. So, the parameter b can assume three values, $b+\delta$, b, and $b-\delta$. We want to show that with these three possible parameter values we are able to direct a trajectory from a starting point P_i to a target located at the vicinity of P_f, by applying N times these perturbations on the b parameter.

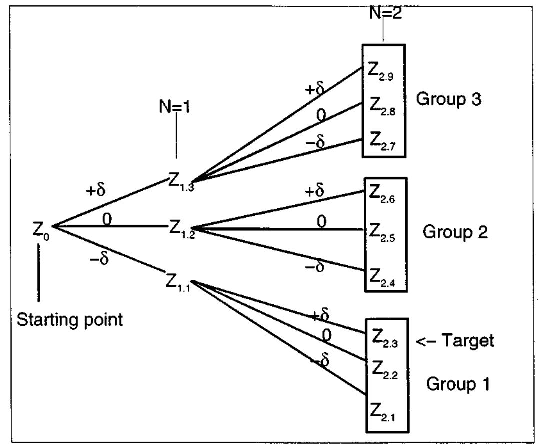

FIG. 1. The path to reach the target, and the value of the constant perturbations $(+\delta,0,-\delta)$ at each iteration. The starting point is indicated by Z_0, and the target by $Z_{2.3}$, which can be achieved by a set of two constant perturbations $(-\delta,+\delta)$.

Thus, to introduce our method, we perform the following procedure. Initially, we apply the map F to the point P_i, using three possible values of the parameter b. So, we get from the first iterate three new points: $Z_{1.1}=F(b+\delta,\ Z_0)$, $Z_{1.2}=F(b,\ Z_0)$, $Z_{1.3}=F(b-\delta,\ Z_0)$. Note that the index $j.k$ indicates the iteration j and the position k of the point among the 3^j points of this jth iteration.

We keep performing this procedure, by applying the map F to each of the three points obtained from the first iterate $(Z_{1.1},\ Z_{1.2},\ Z_{1.3})$, for the three possible parameter values. Thus, at the second iteration we get nine points $(Z_{2.1},\dots,\ Z_{2.9})$. Furthermore, for the jth iteration we can get 3^j points. We stop iterating the map when one of these points reaches the vicinity of P_f, that is, a sphere with radius ϵ whose center is P_f.

A set of perturbations that directs the trajectory from the point P_i to the target is represented in Fig. 1, which indicates a path to reach the target at $Z_{2.3}$, after $N=2$ iterations.

When N is high the amount of memory needed to keep all the points that form the 3^N paths is large. However, in our numerical procedure, we do not need all this information. In fact, at each iteration j, we only need to know the 3^j points obtained at this iteration, to proceed to the next iteration. Or, if the target is reached, we need only to keep the numbers $j=N$ and $k=H$ of the index.

In Fig. 1 we see the paths from the starting point P_i. We put together the three points $(Z_{2.1},\ Z_{2.2},\ Z_{2.3})$ obtained from the point $Z_{1.1}$ and call this set **group 1**. So, **group 2** is formed by the three points $(Z_{2.4},\ Z_{2.5},\ Z_{2.6})$ obtained from the point $Z_{1.2}$ and **group 3** is formed by the other three points. Inside each group, the point with the highest position number k in the index was obtained from the iteration of the former point by applying a positive parameter perturbation $(+\delta)$, and the point that has the lowest k was obtained by applying a negative parameter perturbation $(-\delta)$. Thus, the index of each point gives its position in a vector that stores the 3^N points. Consequently, for N higher than two, the set of applied perturbations S (to reach the target) is easily obtained by knowing only the numbers N and H in the index of the point $Z_{N.H}$ that reaches the target.

Next, we show how to determine the set S. Thus, imagine that the target was reached by the point $Z_{N.H}$ at the Nth iteration and this point belongs to **group M**.

The **group M** (formed at the Nth iteration) is obtained by the iteration of the point $X_{N-1.M}$. This point, depending on the perturbation $\{+\delta,\ 0,\ -\delta\}$, generates the points $(Z_{N.3M-0},\ Z_{N.3M-1},\ Z_{N.3M-2})$ that form **group M**. So, depending on the position of the point $Z_{N.H}$ in **group M** we know the value of the perturbation. For example, if $Z_{N.H} = Z_{N.3M-2}$, we know that the point $Z_{N-1.M}$ was iterated using the perturbation $-\delta$.

To know how the point $Z_{N-1.M}$ was obtained, we have to find out the group L to which this point belongs and its position inside this group. So, in Fig. 1 the set of applied perturbation to the target is $S=\{-\delta,+\delta\}$.

To have a better idea of our method one can compare the results shown in Fig. 1 with the trees calculated to apply the Kostelich's method.[22] These trees contain N-secondary paths, formed by N-backward iterations $Z_{(T-N)}$ of the target $Z_{(T)}$ for different perturbing parameters. If the Kostelich's tree contained infinite secondary paths for infinite different perturbations, it would contain our paths obtained for the three perturbations $+\delta$, 0, and $-\delta$. Therefore, it may be more convenient to apply our method whenever restrictions on the necessary parameter variations are present.

In this section we showed how we can direct a system using three possible perturbations. However, a different number of perturbations can also be used and the previous described derivation also apply.

In this pictorial exposition, we showed one path that directs the starting point to the target. However, more than one path is possible and, in fact, the larger is N the higher is the number of possible paths.

III. TARGETING THE LOGISTIC MAP

The equation of the Logistic map is[26]

$$X_{N+1}=bX_N(1-X_N), \tag{1}$$

where b is the control parameter.

Following the procedure introduced in the previous section, we choose to change the parameter $b=3.78$ by an amount $\delta=0.005$, to reach a target, i.e., the interval $[X-\epsilon,\ X+\epsilon]$ with $\epsilon=0.0001$.

The necessity of the targeting procedure can be seen in Fig. 2, where a map (1) trajectory takes 2406 iterations until the trajectory goes from $X_0=0.330$ to $X_f=0.816$. However, using our method, after only eight iterations, through the set of perturbation $S=\{0,+\delta,\ +\delta,\ 0,+\delta,\ 0,\ 0,\ -\delta\}$, the trajectory reaches the same target located on the point X_f, as shown in Fig. 3. We should emphasize that the perturbed trajectories reach points that are never visited by the attractor of Eq. (1) with constant b, as the point with $X_f=0.750$, which can be reached from $X_0=0.500$ in 10 iterations, for $b=3.78$.

Figure 3 shows the trajectory obtained (solid black line) by applying the determined sequence of perturbations. This trajectory is very close to a stable manifold (dotted gray line) of the target around X_f. This means that the driven trajectory from the starting point to the target is along a stable manifold of the target. Therefore, this stable manifold of the target and the targeting trajectory can hardly be distinguished.

In fact, as pointed out in Ref. 24, the perturbations allow the trajectory to shadow the stable manifold of the target zig-zagging it. Thus, the system is perturbed in such a way that its trajectory either approach the stable manifold, the

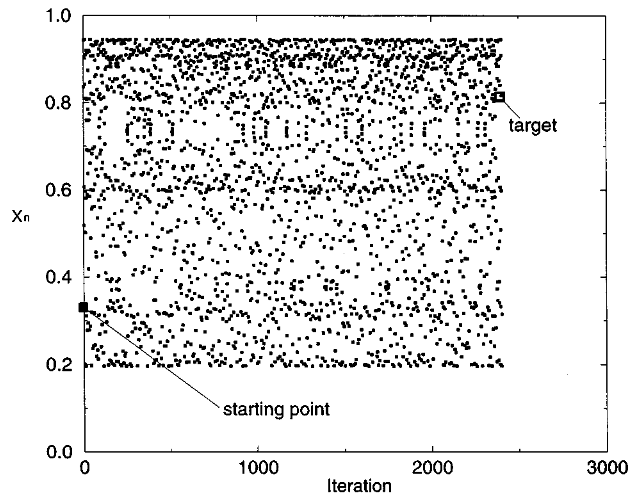

FIG. 2. The time ($n=2446$) that the Logistic map trajectory expends to go from the starting point 0.33 to the target, without applying our targeting method.

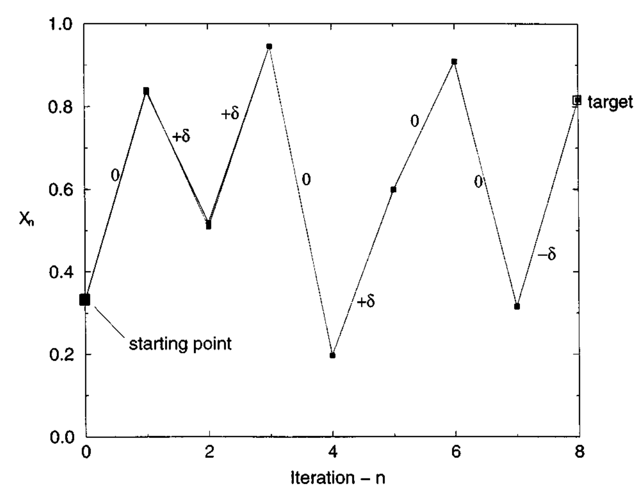

FIG. 3. Using the targeting method we can direct the trajectory from the $X_0=0.33$ to the target positioned at $X_f=0.816$ in only 8 iterations, applying a set of 8 perturbations to the Logistic control parameter $b=3.78$. The amplitude perturbation is $\delta=0.005$. The trajectory (solid black line) is very close to a stable manifold (dotted gray line) of the target.

preferable approach (in this case the perturbation is of the bang–bang type), or get away from it (in this case the perturbation is not of the bang–bang type).

In Fig. 3, we see that the coordinates X_3 and X_6 are close to each other. In this case, one could iterate the X_2 using a special perturbation $+\delta+\Delta$, instead of $+\delta$ (the perturbation indicated in Fig. 3), to obtain $X_3=X_6$. Consequently, the targeting trajectory would reach the target in only five iterations, a shorter time than applying only constant perturbations. The orbit from X_3 to X_6, in Fig. 3, can be considered a pseudo-periodic orbit (the one corresponding to an almost closed loop),[6] and a carefully adjustment in the third perturbation $(+\delta+\Delta)$ could eliminate such a pseudo-periodic orbit reducing even more the targeting time. In Refs. 23,24 the perturbations are computed by using a control optimal algorithm, which probably reduces automatically the targeting time by avoiding pseudo-periodic orbits. This was the procedure used in Ref. 6 to reduce the targeting time.

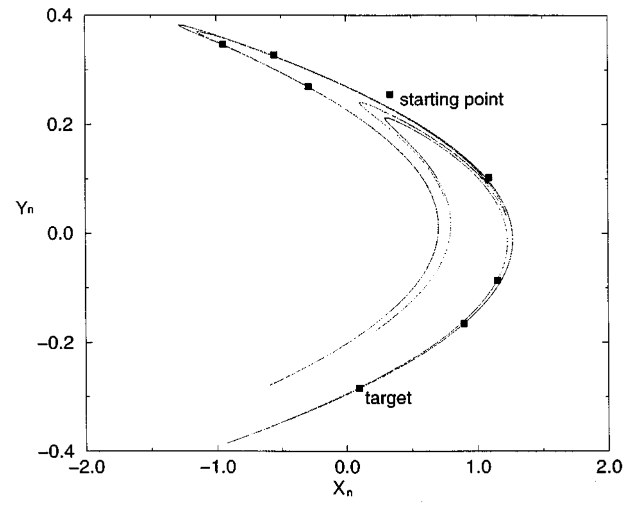

FIG. 4. The targeting method applied to the Hénon map. Without using the targeting 5039 iterations are necessary to direct the starting point to the target. Using the targeting method only 7 iterations are needed.

IV. TARGETING THE HÉNON MAP

The equations of the Hénon map are[29]

$$X_{N+1}=1+Y_n-aX_N^2,$$
$$Y_{N+1}=bX_N. \tag{2}$$

In this case, we consider that the control parameter $a=1.40$ can be changed by an amount of $\delta=0.01$. So, the parameter a can assume the values $a=1.39$, $a=1.40$, and $a=1.41$. Then, one example of the use of our method is considering the starting point $P_i=\{0.3409,\ 0.2546\}$ and the target at the point $P_f=\{0.0979,\ -0.2846\}$. 5039 iterations are necessary for the trajectory to go from the starting point P_i to the vicinity of P_f, i.e., to reach a sphere with radius $\epsilon=0.001$ whose center is P_f. However, with our method, the trajectory reaches the target in only 7 iterations, by iterating Eq. (2) with the following set of control perturbations: $S=\{+\delta,-\delta,+\delta,-\delta,+\delta,\ 0,-\delta\}$. This example can be seen in Fig. 4. The same initial and final points (in a rescaled different version of the Hénon map) were used in Ref. 3, and in that work 12 iterations were necessary to direct the initial point to the target.

Using the same Hénon map, in Ref. 23 the authors target the trajectory from $P_i=\{0.60000,\ 0.20000\}$ to $P_f=\{0.63135,\ 0.18941\}$ with a precision ϵ and a perturbation bounded to a maximum value μ. In this paper the authors propose an optimal control approach to reach the target either fast targeting or choosing the best precision target.

The fastest targeting presented in Ref. 23, for $\epsilon=0.02$ and $\mu=0.04$, directs the trajectory from the point P_i in only 3 iterations. If we apply our method for $\epsilon=0.02$ and $\delta=0.04$, we are able to reach the target in 9 iterations. However, for $\delta=0.1$ we can reach the target in only 2 iterations with the perturbing set $S=\{\delta,-\delta\}$.

On the other hand, in Ref. 23 the authors have directed from the same point P_i, with $\epsilon=10^{-5}$ and $\mu=0.01$, in 11 iterations. Applying our method for $\epsilon=10^{-5}$, $\delta=0.01$, and three different perturbations, we can reach the target in 13 iterations. However, if we consider $\delta=0.02$ we can also reach the target in 11 iterations with the perturbing set $S=\{-\delta,-\delta,\ 0,-\delta,\ 0,\ 0,\ \delta,-\delta,-\delta,-\delta,-\delta\}$.

Furthermore, while in Ref. 23 the same target is reached, for $\epsilon=0.002$ and $\mu=0.005$, in 15 iterations, for the same ϵ and δ we reach the target in 10 iterations. Besides, we find 3096 different ways to reach the target from the point P_i; one of these samples is $S=\{-\delta,-\delta,-\delta,-\delta,-\delta,\ \delta,\ \delta,\ \delta,-\delta,\ \delta\}$.

Due to the ergodic property of strange attractors we can always reach any target in these attractors. In a general way, the smaller is δ the slower is the time to reach the target. In conclusion, for this map our method works as well as the previous one.

A. Creating and controlling orbits

It is also possible to use the targeting method to create unstable periodic orbits. To do so, we choose the starting point as the target. Since the starting point does not coincide exactly with the final point, and the metric distance between these two points are ϵ bounded, this trajectory is an almost closed cycle. Moreover this cycle does not exist in the non perturbed attractor, but it is rather a new trajectory. In addition, this trajectory is unstable, since the reapplication of the previous set S or the presence of a small noise level would take the system trajectory away from this aimed cycle. Even so the system can be trapped along this cycle just computing a new set S each time the system trajectory reaches the target.

Thus, we apply our target method to direct the trajectory from the point $P_i=\{0.2798,-0.2606\}$ to the point $P_f=P_i$ with a precision $\epsilon=0.001$ and an amplitude $\delta=0.01$. To target this point we need to apply eleven perturbations on the parameter a: $S=\{-\delta,-\delta,\ 0,-\delta,\ 0,\ 0,-\delta,-\delta,-\delta,+\delta,+\delta\}$. This period-11 orbit can be seen in Fig. 5.

To trap the trajectory of Eq. (2) to this period-11 orbit we have to apply a new set of perturbations for each cycle,

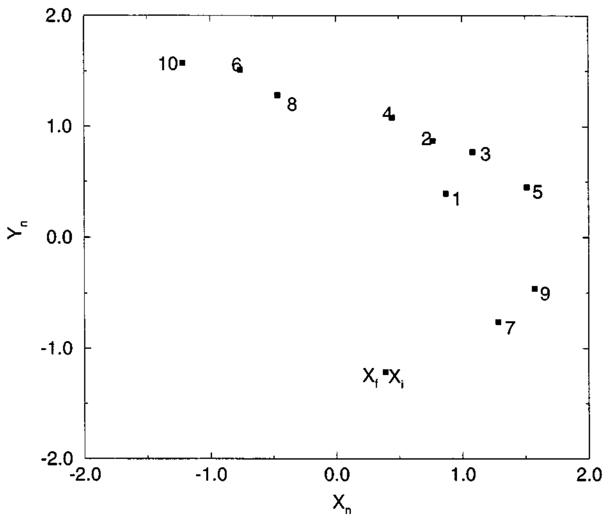

FIG. 5. The new period-11 of the unstable periodic orbit created by applying our targeting method to the system (2).

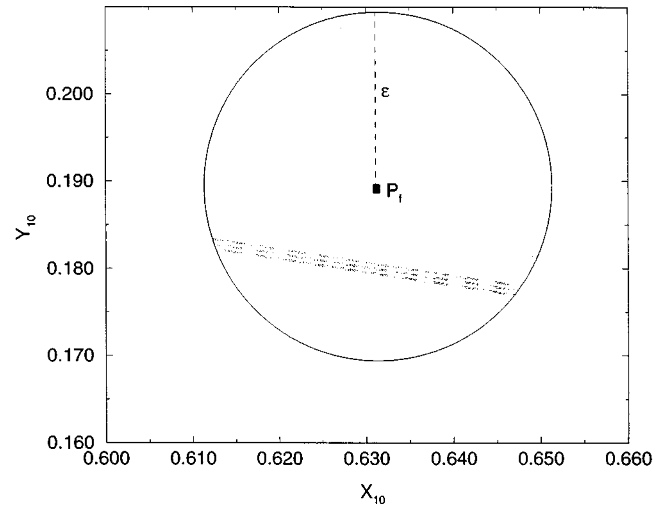

FIG. 7. The 1767 targeted trajectories and the target enclosed by a circumference positioned at P_f and radius ϵ.

since this orbit is unstable. Thus, in Fig. 6, after targeting from a generic point $P_i = \{0.734362,\ 0.163519\}$ to the vicinity of the point $P_f = \{0.2798,\ -0.2606\}$ which takes 12 iterations, we start applying our targeting method to make the trajectory of Eq. (2) shadow the almost close period-11 orbit shown in Fig. 5.

Another way for stabilizing this orbit could be by applying the OGY method,[9] adapted to the control of this long period orbit (see Ref. 37). With this method one could make fine corrections to the sequence of computed perturbations S (to direct the point P_i to P_f), in order to stabilize the almost closed cycle.

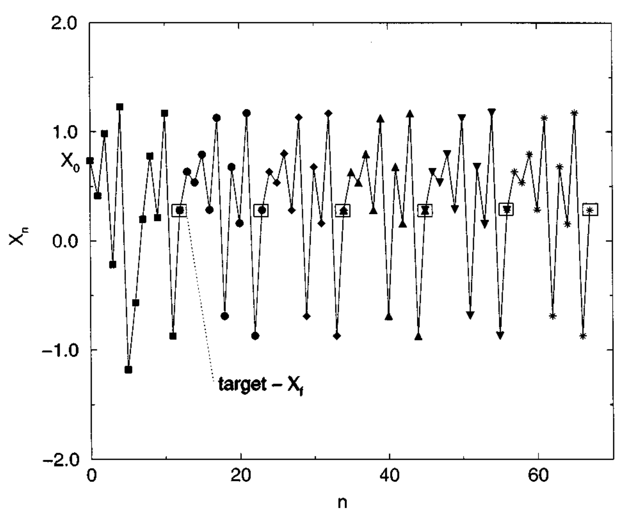

FIG. 6. Controlling the orbit of Fig. 5 by applying five sucessive sets of perturbations, in order to maintain the trajectory close to the period-11 orbit, after the target is reached in 12 iterations.

B. Noise

To study how our method is noise sensitive, we apply Eq. (2) to calculate exact sets of perturbations to targeting and, after that, these sets are applied in the presence of noise.

Thus, considering the same $P_i = \{0.60000,\ 0.20000\}$, $P_f = \{0.63135,\ 0.18941\}$ used before, we find 1767 different ways to reach the target in absence of noise, with 10 iterations and $\epsilon = 0.02$. So, in Fig. 7, we see all these perturbed trajectories that leave the point P_i and go to the target, the area inside the circumference with radius ϵ and center at P_f. We see in this figure that the targeted points are at least 0.008 away from the point P_f.

Next, we apply these 1767 possible sets of perturbations into the Hénon Map, considering that the perturbing set (computed in the absence of noise) may suffer changes due to uniformly distributed random dynamical noise. So, after adding noise, we analyze the performance of directing the starting point, verifying if the target can still be reached. This is done analyzing the spatial distance between the directed points and the point P_f.

In Fig. 8 we apply noise with a randomic variation of 3% in the δ value. For that, 158 of the 1767 directed trajectories are outside the target region limited by the dashed line. In addition, though these points are not within the target, they are close to it. If we increase the noise to 5%, 273 directed trajectories go outside the target (Fig. 9). Both Figs. 8 and 9 show that the minimum distance between the targeted points and P_f is the same than when we do not apply noise.

To see how much noise magnitudes affect our targeting method we show in Fig. 10 the number of trajectories that failed to reach the target with respect to the noise magnitude. We see that for noise magnitudes until 26% the number of failed trajectories increases linearly, and for higher magnitudes there is an inverse exponential rate of convergence to the total number 1767 of targeting trajectories. Therefore, our method is robust even for high noise amplitudes.

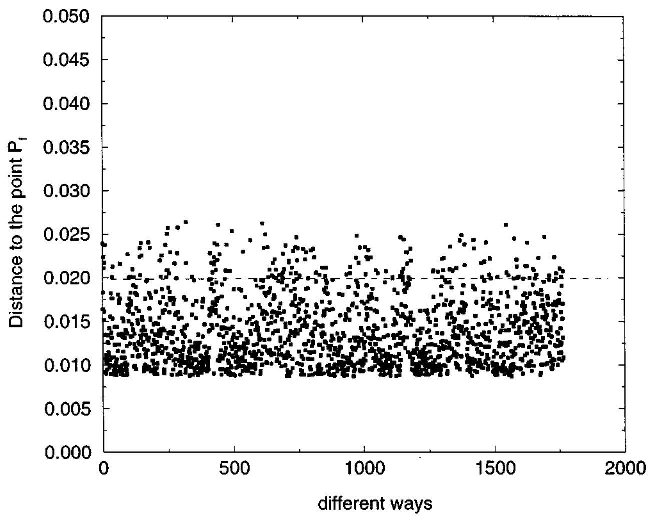

FIG. 8. The 1767 targeting points when 3% of magnitude noise is added to the set of perturbations. In this case 158 points are above the target region delimited by the dashed line.

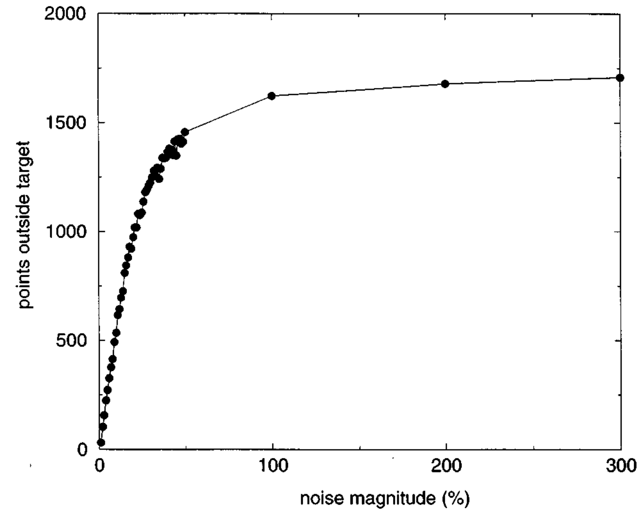

FIG. 10. Number of exit trajectories (trajectories that do not reach the target) with respect to noise magnitude.

V. TARGETING FLUXES—THE DOUBLE SCROLL CIRCUIT

The Double Scroll circuit (see Fig. 11) is an autonomous nonlinear electronic circuit composed by two capacitors, C_1 and C_2, one inductor, L, two linear resistor, R, r, and a nonlinear resistor R_{NL}. The circuit dynamic is described by

$$\begin{aligned} C_1 d_t V_{C1} &= (V_{C2} - V_{C1})/R - i_{NR}(V_{c1}), \\ C_2 d_t V_{C2} &= (V_{C1} - V_{C2})/R + i_L, \\ L d_t i_L &= -V_{C2} - q(t), \end{aligned} \tag{3}$$

where V_{C1}, V_{C2}, and i_L are the dynamical variables and represent the voltage across C_1, the voltage across C_2, and the current through L, respectively, and q is the voltage across r, which is the external perturbation we consider to apply our targeting method.

The term i_{NR} is the characteristic curve of the nonlinear resistor R_{NL} and is the piecewise-linear function represented by the equation

$$i_{NR} = m_0 V_{C1} + \tfrac{1}{2}(m_1 - m_0)(|V_{C1} + B_p| - |V_{C1} - B_p|). \tag{4}$$

Equations (3) were integrated using the following parameters:

$$\begin{aligned} &\frac{1}{C_1} = 9.0, \quad \frac{1}{C_2} = 1.0, \quad \frac{1}{L} = 7.0, \\ &\frac{1}{R} = 0.7, \quad m_0 = -0.5, \quad m_1 = -0.8, \quad B_p = 1.0. \end{aligned} \tag{5}$$

Figure 12 shows the Double Scroll chaotic attractor of (3) for $q = 0$.

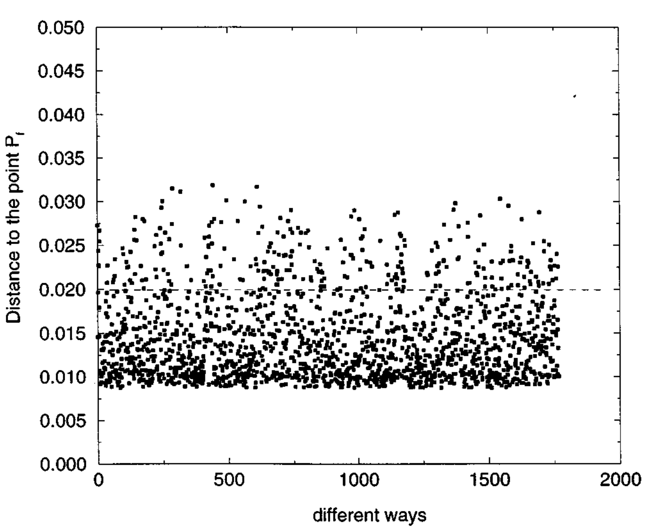

FIG. 9. The 1767 targeting points when 5% of magnitude noise is added to the set of perturbations. In this case 273 points are above the target region delimited by the dashed line.

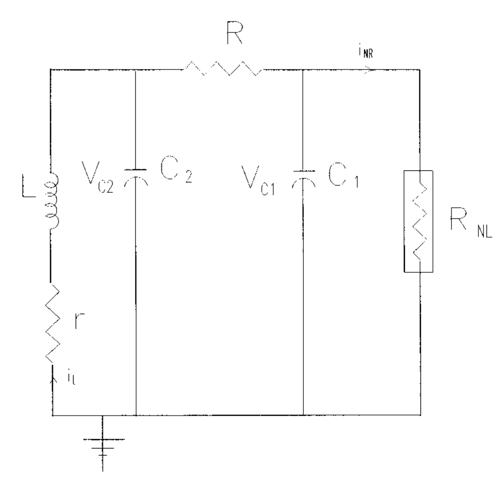

FIG. 11. The Double Scroll circuit composed by two capacitors, C_1 and C_2, one inductor, L, two linear resistors, R and r, and one non-linear resistor R_{NL}.

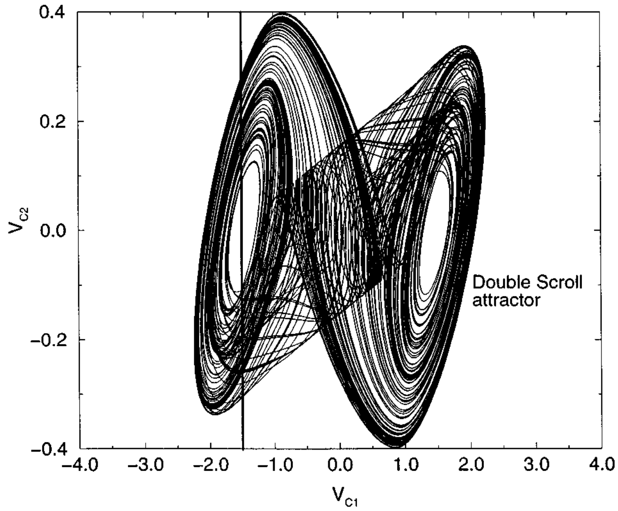

FIG. 12. The nonperturbed Double Scroll attractor of the system (3) projected on the variables V_{C1} and V_{C2}. The line represents the chosen Poincaré section we consider to apply our targeting method.

In the chosen example, we want to direct the trajectory of Eq. (3) to the target located at the point

$$V_{C1}^{f}=-1.500, \quad V_{C2}^{f}=0.238, \quad i_L^{f}=1.723, \tag{6}$$

with a precision of $\epsilon_f=0.001$.

We consider that both the starting point and the target, are on a Poincaré surface α,[20] at the plane determined by $V_{C1}=-1.5$ (see the line in Fig. 12).

To apply our method to a three-dimensional (3D) flow, some adjustments to the original method are necessary. For a map, the perturbation is introduced at each iteration, yet for a flow we choose a convenient time interval T, during which the control perturbation, q, is applied. During the target, the trajectory crosses the section α J times. As in previous sections, we consider the number of times, N, we apply the constant perturbation, q, which as before assume three possible values: $+\delta$, 0, $-\delta$.

Intending to simulate a real experiment, we do not set up the system initial condition as it can be done for a system of known equations. So, before starting to apply our method, we force the trajectory to oscillate in a periodic way. The advantage of this procedure is the facility of determining the starting point, since a crossing on the Poincaré section of any periodic orbit can be chosen as our starting point.

So, initially, we choose a perturbation $q(t)$ that forces the circuit to oscillate periodically. This is done by a sinusoidal wave of amplitude c and frequency f:

$$q(t)=c\ \sin(2\pi f t). \tag{7}$$

It is possible to suppress chaotic motion described by Eq. (3) by perturbing it with q given by Eq. (7) (chaos is suppressed by phase-locking).[15–17,38] Then, we choose the frequency $f=0.3$ and amplitude $c=0.022$ to make Eq. (3) behave periodically.

This point chosen as the starting point P_i is

$$V_{C1}^{i}=-1.5000, \quad V_{C2}^{i}=0.3361, \quad i_L^{i}=2.0289. \tag{8}$$

So, we perturb the trajectory until it is in the vicinity of the starting point, i.e., when the trajectory reaches a three-dimensional sphere with center in the point P_i and radius $e_i=0.0005$. When that happens, we start applying our method for a set of tentative parameters (T, N, δ) previously chosen, and the perturbation is not anymore given by Eq. (7), but is rather a series of constants δ as introduced before.

Now, we estimate the number N of perturbations we apply into the system. To do that, we consider the area the attractor occupies on the Poincaré section. We find this area (the maximum length times the maximum width of the attractor on the section) to be $A\approx 0.0091$. Thus, the minimum number N is found by

$$\frac{A}{3^N}<\epsilon_f^2, \tag{9}$$

leading us to $N=8$.

Now, we choose the time interval, T, during which we apply each perturbations q. For that, we verify that the approximated time interval the trajectory, starting from the point P_i, spend to return to the chosen Poincaré section is $\tau=16$. Then, the time interval T is obtained from

$$T=\frac{\tau}{N}, \tag{10}$$

which gives us $T=2$. Therefore, in this case the perturbation is applied $N=8$ times during a Poincaré mapping cycle.

In this work we choose $\delta=c=0.022$, that is, the sinusoidal wave (in the phase-locking) and the targeting perturbation have the same amplitude. As a matter of fact, the success of the targeting does not depend much on the amplitude δ.

Summarizing the application of the targeting method to a flux we first stabilize the system until the trajectory crosses the point P_i, by applying a sinusoidal wave (the phase-locking targeting phase). Then we apply N perturbations, each one during a time T. After that, we still keep integrating until the trajectory reaches the target on the Poincaré section. If so, we consider J as the number of times the trajectory crosses the section when the perturbations are applied.

For $T=2$, $N=8$, and $\delta=0.022$, we found that the target can be reached by applying the following two sets of perturbations: $\{-\delta$, -δ, 0, 0,$-\delta$, 0, 0, $\delta\}$ with $J=5$ and $\{+\delta$, $-\delta,+\delta,+\delta$, 0,$-\delta,-\delta,-\delta\}$ with $J=6$. For a high number of perturbations, we find many ways to reach the target.

However, we can use other parameters instead of the ones we have estimated. Thus if we consider the same $N=8$, $\delta=0.0022$ but $T=1.2$, we find the following perturbation set that directs the system to the target:

$$\{-\delta,\ 0,-\delta,-\delta,-\delta,-\delta,+\delta,-\delta\}, \tag{11}$$

with $J=2$. Figure 13 shows the resulting trajectory for this set.

VI. APPLICATION OF THE TARGETING PROCEDURE

The trajectory of Eqs. (3) can be controlled by applying a perturbation q given by Eq. (7) (Ref. 38). But, there are other ways of doing that, as using the OGY[9] method. With

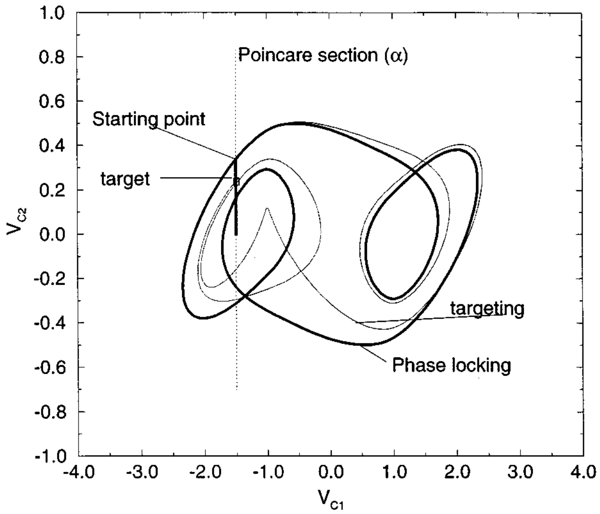

FIG. 13. Application of the targeting method to the Double Scroll circuit. The starting point is reached by phase-locking the trajectory through an external sinusoidal perturbation. After that, the trajectory is target by applying 8 perturbations. On the surface α is indicated in the region we can reach from the starting point applying different sets of 8 perturbations.

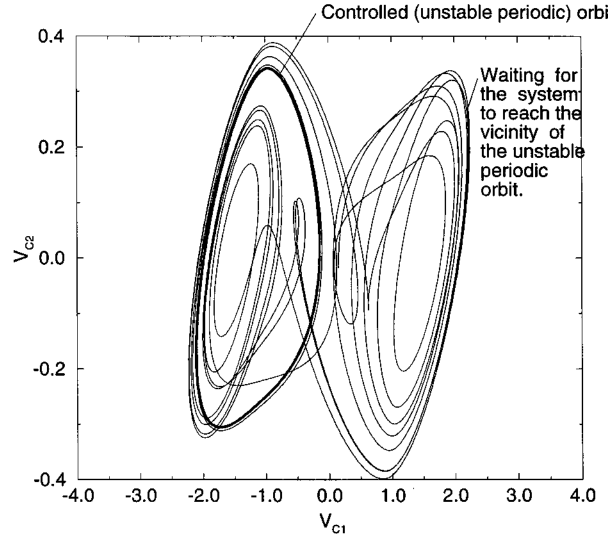

FIG. 15. The stabilization of the unstable periodic orbit by applying the OGY method, after the trajectory reaches the vicinity of the unstable periodic orbit.

this method it is possible to control a chosen unstable periodic orbit, as the one that can be seen in Fig. 14. However, this method requires the system trajectory to get closer to the periodic orbit to be controlled. So, we can use our targeting method to rapidly direct the trajectory of Eq. (3) to a point near the chosen unstable periodic orbit. After we reach the vicinity of this orbit, we apply the OGY method to stabilize it. The control of the unstable periodic orbit (Fig. 14) by the OGY method can be seen in Fig. 15.

This orbit was controlled by applying small perturbations on the parameter q. Thus, after the trajectory of Eq. (3) reached the vicinity of the unstable periodic orbit, on the Poincaré section $V_{C1}=-1.5$, we vary q by δq, according to the equation

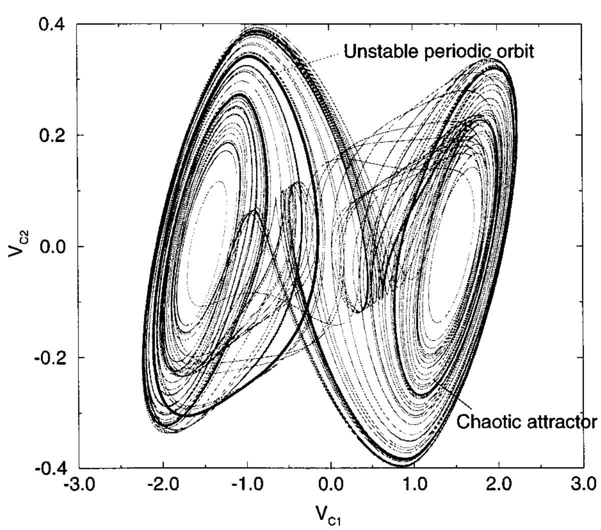

FIG. 14. The non perturbed Double Scroll attractor and one of the infinite unstable periodic orbits embedded in the attractor.

$$\delta q=(0.1649,\ 0.1818)(\xi_n-Z_f), \tag{12}$$

where ξ represents the trajectory position (a vector representing the variables V_{C2} and i_L) when it crosses the Poincaré section, and Z_f represents the V_{C2} and i_L coordinates of the target. So, each time the trajectory crosses the section we change the value of the parameter q by using Eq. (12). A detailed manner of obtaining the formula (12) can be found in Ref. 9.

The point given by Eq. (6) was chosen as the target because it is in the vicinity of the crossing between the unstable periodic orbit (shown in Fig. 14) and the section α. So, we can use our method to direct the trajectory of Eq. (3) rapidly to the vicinity of this orbit, and then apply the OGY method. The result is shown in Fig. 16.

In this figure we first apply the sinusoidal perturbation to induce the phase-locking. Then, when the trajectories reaches the starting point we apply our method to direct it to the target [see Eq. (6)] (large black line). When the target is reached, we apply the OGY method to control the unstable periodic orbit (gray line).

In Fig. 17 we show this control using the time evolution of the variable $V_{C1}(t)$, indicating all the phases: the phase-locking, the targeting procedure, and the control of the unstable periodic orbit.

VII. CONCLUSIONS

In this work we presented a new method, for targeting a chaotic trajectory, that uses only a sequence formed by few different parameter perturbations. A large amount of memory is not needed to determine this sequence of perturbations. In fact, after applying this method, it is only necessary to keep the number of required iterations and the relative position of the targeting trajectory, among all the considered paths, obtained at the last iteration. Therefore,

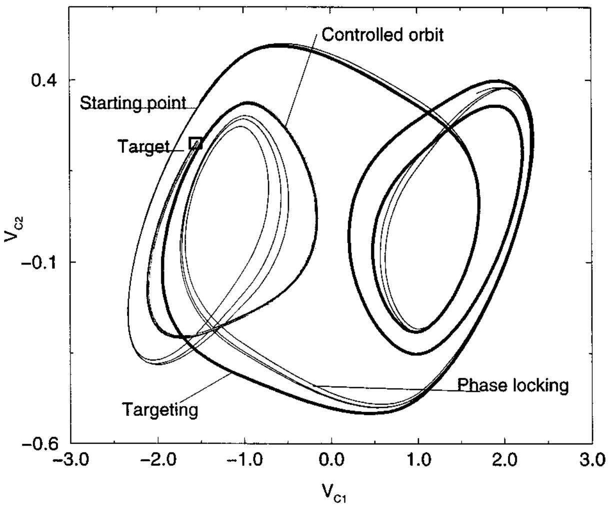

FIG. 16. The targeting method is used to rapidly direct the trajectory to the vicinity of the unstable periodic orbit. We apply our targeting method to direct the trajectory to the target, located in the vicinity of an unstable periodic orbit. Then, we apply the OGY method, stabilizing this orbit.

with our method only two integer numbers are enough to reconstruct the determined targeting trajectory.

The chosen perturbation sequence for targeting may not be unique. For any of these sequences the trajectory evolves along a stable manifold of the target.

With this method we showed that the Logistic and Hénon maps, and the Double Scroll circuit can have their trajectories rapidly directed, from a starting point to a chosen target. For that, only three possible parameter perturbations were considered, but we also obtained similar results applying more than three different perturbations. Therefore, although the number of different perturbations is not so critical, it is convenient to use a small number of them.

With the targeting method we created new unstable periodic orbits that were not present in the attractor of the nonperturbed system. These new periodic orbits are unstable; however, they were stabilized by applying successive sets of perturbations, determined by our targeting method. Even in the presence of noise, this stabilization is possible since our target method is very noise robust as shown for the Hénon map.

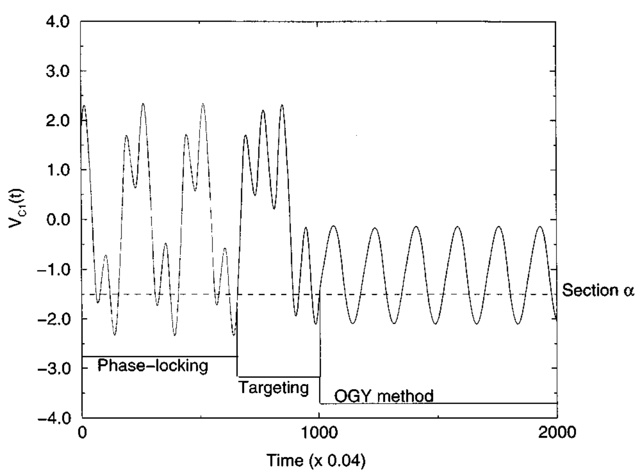

FIG. 17. Targeting the system (3) and then controlling the unstable periodic orbit (shown in Fig. 14) by applying the OGY method.

To simulate the application of our method to a real experiment, we neither set up the system initial conditions (not regularly achievable) nor work with a low-dimension mapping, as used in other targeting approaches. Instead, we achieve the starting point forcing the system with an external phase-locking sinusoidal perturbation. Once a convenient starting point is reached, we can apply our targeting method to direct the trajectory to the desired target.

To illustrate a practical application of our target method we rapidly directed a trajectory of the Double Scroll circuit to a point located at the vicinity of a chosen unstable periodic orbit, that afterword was stabilized by applying the OGY control method.

Although we presented results obtained with low-dimensional systems (only one positive Lyapunov coefficient) preliminary results show that the method can be successfully applied to high-dimensional systems as the Double Kicked Rotor (two positive Lyapunov coefficients and an estimated Lyapunov dimension of 2.84).[37]

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