Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physica D 212 (2005) 216–232
PHYSICA D
www.elsevier.com/locate/physd

Non-transitive maps in phase synchronization

M.S. Baptista [a,b,*], T. Pereira [a,b], J.C. Sartorelli [b], I.L. Caldas [b], J. Kurths [a]

[a] *Universität Potsdam, Institut für Physik Am Neuen Palais 10, D-14469 Potsdam, Deutschland*
[b] *Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, SP, Brazil*

Received 4 July 2005; received in revised form 21 September 2005; accepted 3 October 2005
Available online 14 November 2005
Communicated by R. Roy

Abstract

Concepts from Ergodic Theory are used to describe the existence of special non-transitive maps in attractors of phase synchronous chaotic oscillators. In particular, it is shown that, for a class of phase-coherent oscillators, these special maps imply phase synchronization. We illustrate these ideas in the sinusoidally forced Chua's circuit and two coupled Rössler oscillators. Furthermore, these results are extended to other coupled chaotic systems. In addition, a phase for a chaotic attractor is defined from the tangent vector of the flow. Finally, it is discussed how these maps can be used for the real-time detection of phase synchronization in experimental systems.

PACS: 05.45.-a; 0.5.45.Xt; 05.45.-r; 02.45.Ac

Keywords: Chaotic phase synchronization; Ergodic Theory; Temporal mappings

1. Introduction

Coupled chaotic systems have recently attracted much attention due to the verification that they may be useful for understanding natural systems in a variety of fields such as ecology [1], neuroscience [2,3], economics [4], and lasers [5,6]. It has been verified that, despite the higher dimensionality of a coupled chaotic system, the coupling among the elements might make them synchronize [7,8], reducing the dynamics of the system to a few degrees of freedom.

In this work, we focus our attention on the phenomenon of Phase Synchronization (PS), which describes the appearance of phase synchronous behavior between two interacting chaotic systems [9], i.e. given two chaotic

* Corresponding author at: Universität Potsdam, Institut für Physik Am Neuen Palais 10, D-14469 Potsdam, Deutschland. Tel.: +49 30 1977 1432; fax: +49 30 1977 1142.
E-mail address: murilo@agnld.uni-potsdam.de (M.S. Baptista).

0167-2789/$ - see front matter
doi:10.1016/j.physd.2005.10.003

systems, their phase difference remains bounded despite the fact that their amplitudes may be uncorrelated. This phenomenon is particularly interesting, since it can arise from a very small coupling strength. Its presence was reported in a variety of experimental systems. It was demonstrated experimentally in electronic circuits [10], and later in electrochemical oscillators [11]. It was found in plasma [12] and in Chua's circuit [13]. Evidence of phase synchronization in communication processes in the human brain was also found [14,15].

To detect PS in a real-time experiment, one has to measure the phase of the chaotic trajectory [16]. However, the phase is not always easily accessible information. To overcome this difficulty, it is important to understand fundamental properties of phase synchronous systems that can easily be verified experimentally. For chaotic systems that are phase synchronized with periodic forcing [17], it was reported that a stroboscopic map of the trajectory was a subset of this attractor and only partially occupies the region delimited by a projection of the attractor. This property was used to detect, in a real-time experiment, phase synchronization between Chua's circuit and a sinusoidal forcing [13].

This approach of detecting phase synchronization through the stroboscopic map can be extended for coupled chaotic oscillators in a formal way. The stroboscopic map is generalized to the **conditional Poincaré map**. Given two oscillators, at least one being chaotic, the conditional Poincaré map is constructed by collecting points in one oscillator at the moment at which some event occurs in the other oscillator. If the set of discrete points generated by this conditional map does not visit any arbitrary region of a special projection of the chaotic attractor, we call this set a **P-set**. This property of the conditional Poincaré map is called non-transitivity [18], i.e. an initial condition under the conditional Poincaré map does not visit everywhere in a subspace of the attractor. Like the stroboscopic maps of oscillators that are in phase synchrony with a forcing, the conditional Poincaré maps of coupled chaotic oscillators, in PS, also only partially occupy a projection of the attractor.

In this work, we show how the conditional Poincaré map can be used to detect PS without actually having to measure the phase. For phase-coherent oscillators, a special type of P-set, which we call PS-set (Phase Synchronization set), exists. Conversely, its existence also implies PS. We illustrate our findings and ideas with numerical and experimental analyses in the forced Chua circuit and the coupled Rösller oscillator [19].

Further, we extend these results to non-phase-coherent attractors. Finally, we also introduce a phase of a chaotic trajectory to be a quantity related to the amount of rotation of the tangent vector. This definition can be applied to chaotic attractors, independently of whether they have phase-coherent or non-phase-coherent dynamics.

This work is organized as follows. In Section 2, we define a way of measuring the phase of a chaotic flow, and discuss the relationship between the average return time and the average angular frequency. In Section 3, we discuss the conditions for PS states and, in Section 4, we describe the phenomenon of PS in the forced Chua circuit. We introduce the notion of a conditional Poincaré map in Section 5 and the P-sets (as well as the PS-sets) in Section 6. In Section 7, we show how PS can be found by detecting these sets in the forced Chua's circuit and, in Section 8, in the coupled Rössler oscillator. Further, in Section 9, we discuss the extension of these ideas to a class of non-coherent oscillators. In Section 10, we give some remarks and the conclusions of this work. In Appendix A, we formally introduce the conditional Poincaré map and the P-set, and in Appendix B we show that, for coherent dynamics, the PS-sets exist if, and only if, there is PS. In other words, PS implies PS-sets and vice-versa.

2. Phase, frequency and average return time of a chaotic attractor

The phase of a chaotic attractor in a projection j (a subspace) is defined as the amount of rotation of the tangent vector in this projection, and can be represented by an integral function of the type

$$\phi_j(t) = \int_0^t \left| \frac{\mathrm{d}\theta(t')}{\mathrm{d}t'} \right| \mathrm{d}t' \tag{1}$$

with $\mathrm{d}\theta(t)$ being an infinitesimal displacement of the tangent vector of the flow, from time t to time $t + \mathrm{d}t$, and $\mathrm{d}t \to 0$. Note that, in Eq. (1), we are measuring the amount of rotation, per unit time, of a projection of the tangent vector of the flow, on the same subspace j where the phase is defined. We call this subspace $\mathcal{P}_j$. The attractor $\mathcal{X}$, projected on the subspace $\mathcal{P}_j$, is regarded as $\mathcal{X}_j$. The instantaneous angular frequency of the trajectory in $\mathcal{X}_j$, named W_j, is given by $\frac{d\phi_j}{dt}$. So, from Eq. (1), $W_j = \left|\frac{d\theta}{dt}\right|$, and the average angular frequency $\langle W_j \rangle$ is

$$\langle W_j \rangle = \lim_{t\to\infty} \left\langle \frac{d\phi_j}{dt} \right\rangle, \tag{2}$$

where $\langle - \rangle$ represents the average. Eq. (2) can be put into the form $\langle W_j \rangle = \frac{\phi_j(t)}{t}$.

Whenever a Poincaré section can be defined, the average period of the chaotic attractor on the subspace $\mathcal{P}_j$ is calculated by

$$\langle T_j \rangle = \frac{\sum_{i=0}^{N} \Delta\tau_j^i}{N}, \tag{3}$$

where $\Delta\tau_j^i = \tau_j^i - \tau_j^{i-1}$, and τ_j^i represents the time at which the trajectory in the subspace $\mathcal{P}_j$ makes the i-th crossing with this Poincaré section.

We introduce $\langle \Delta\phi_j \rangle$ to be the average displacement of the phase for a typical period as

$$\langle \Delta\phi_j \rangle = \frac{\phi_j(N)}{N}, \tag{4}$$

with $\phi_j(N)$ being the phase associated with the subspace $\mathcal{P}_j$ at the moment that the N-th crossing between the trajectory and the Poincaré section occurs.

Thus, we can write Eq. (2) as

$$\langle W_j \rangle = \frac{\langle \Delta\phi_j \rangle}{\langle T_j \rangle}. \tag{5}$$

For the forced Chua circuit, the subspace $\mathcal{P}_1$ is defined by a suitable projection of the circuit variable. We have that $\langle \Delta\phi_1 \rangle = 2\pi$. So, Eq. (5) can be written as $\langle W_1 \rangle = \frac{2\pi}{\langle T_1 \rangle}$. For the coupled Rössler oscillator, the quantities in Eq. (5) can be calculated in two subspaces: the subspace $\mathcal{P}_1$ associated with the variables of one Rössler system and the subspace $\mathcal{P}_2$ associated with the variables of the other Rössler system. As shown in [21], $\langle \Delta\phi_1 \rangle$ might slightly differ from 2π, and thus $\langle W_j \rangle = \frac{\langle \Delta\phi_j \rangle}{\langle T_j \rangle}$.

So, Eqs. (2) and (5) relate the average period, the average angular frequency, and the phase of a chaotic trajectory. This shows that the average period (recurrence) and the average angular frequency are intimately connected in phase-coherent chaotic systems, and both these quantities can be calculated from the phase.

3. Phase synchronization

Having defined phase, PS exists whenever the following condition is satisfied:

$$|\phi_1(t) - r\phi_2(t)| < \langle \Delta\phi_1 \rangle. \tag{6}$$

The minimal bound for the phase difference, $\langle \Delta\phi_1 \rangle$, in terms of the phase, as defined by Eq. (6), was estimated theoretically in Ref. [21]. Eq. (6) means that the phase difference between the two coupled systems is always bounded, and r is a rational constant [22].

Also,

$$\langle W_1 \rangle - r\langle W_2 \rangle = 0. \tag{7}$$

In this work, we will consider cases for $r = 1$. Otherwise, a simple change of variables could eliminate this constant from Eq. (7). For the forced Chua circuit $\langle W_2 \rangle = \omega$, with $\omega = 2\pi f$ representing the angular frequency of the forcing. There is PS if $\langle W_1 \rangle = \omega$. Therefore, $\langle T_1 \rangle = 1/f$. For the coupled Rössler system, if PS exists, then $\langle W_1 \rangle = \langle W_2 \rangle$, $\langle R_1 \rangle = \langle R_2 \rangle$, and $\langle \Delta\phi_1 \rangle = \langle \Delta\phi_2 \rangle$, and therefore, we could have $\langle \Delta\phi_2 \rangle$ in the right-hand term of Eq. (6) instead of $\langle \Delta\phi_1 \rangle$.

4. The sinusoidally forced Chua circuit

The circuit is represented by:

$$C_1 \frac{\mathrm{d}X_1}{\mathrm{d}t} = g(X_2 - X_1) - i_{NL} \tag{8}$$

$$C_2 \frac{\mathrm{d}X_2}{\mathrm{d}t} = g(X_1 - X_2) + X_3 \tag{9}$$

$$L \frac{\mathrm{d}X_3}{\mathrm{d}t} = -X_2 - V \sin(\omega t) \tag{10}$$

where X_1, X_2, and X_3 represent, respectively, the tension across two capacitors and the current through the inductor (see [13] for more details), and ω and V are the angular frequency and the amplitude of the forcing, respectively. The piecewise-linear function i_{NL} is given by:

$$i_{NL} = m_0 X_1 + 0.5(m_1 - m_0)[|X_1 + B_p| - |X_1 - B_p|] \tag{11}$$

where we have chosen the parameters $C_1 = 0.1$, $g = 0.574$, $C_2 = 1$, $L = 1/6$, $m_0 = -0.5$, and $m_1 = -0.8$, such that we obtain a Rössler-type attractor, for $V = 0$.

To calculate the phase of the chaotic trajectory, we first define the subspace $\mathcal{P}_1$ to be given by the pair of variables (X_1, X_2), and then we use Eq. (1). In Fig. 1, we show the difference between the phase of the chaotic circuit (as calculated using Eq. (1)) and the phase of the forcing, ωt. In (a), the phase difference is bounded and the average period of the chaotic attractor, defined as the average recurrence time of trajectories that cross the section $X_2 = 0$, is equal to $\langle T_1 \rangle = 3.57015$, which is equal to $1/f$, since $f = 0.2801$. The average angular frequency can be calculated using Eq. (5), which gives us $\langle W_1 \rangle = 1.75992$. Or, from Eq. (2), we have $\langle W_1 \rangle = 1.75992$. Note that the average growth of the phase (calculated using Eq. (1)) for a typical average period is 6.28318..., which is 2π. In Fig. 1(b), we have that $\langle T_1 \rangle = 3.57006$, which is different from $1/f$, since $f = 0.279$. So, in (b) there is no PS, and consequently inequality (6) is not satisfied.

In Ref. [13] we detected PS experimentally in the forced Chua circuit whenever stroboscopic maps could be constructed for a time interval equal to $\Delta\tau_1 = \frac{1}{f}$ such that this map, projected into the same subspace considered to calculate the phase, does not occupy the region occupied by the attractor projected into the same subspace.

To understand this technique, we assume that $\phi_1(t)$, the phase of the chaotic trajectory, is the angle (on the lift) described by the vector position of this trajectory (Rössler-like attractor) and $\phi_2(t) = \omega t$ is the phase of the forcing. If Eq. (6) is satisfied at any time, then it is satisfied at multiples of the period of the forcing, $\tau_1^i = \frac{i}{f}$. So, we get $|\phi_1(\tau_1^i) - 2\pi i| < 2\pi$, which means that a stroboscopic map has to be concentrated in an angular section smaller than 2π. The stroboscopic map, which is already a subset of the chaotic flow, projected into the same subspace considered to calculate the phase, does not occupy the whole region occupied by the attractor projected into this same subspace. Another property of the stroboscopic map is that points in it are mapped into it by looking at the trajectory after a time interval given by $\Delta\tau_1$, so it is a subset that is recurrent to itself.

Using this technique, and for the same parameters as [13], we show in Fig. 2(a) the experimental synchronization region for the forced Chua circuit in the parameter space $V \times f$. The triangular shaped region represents parameters

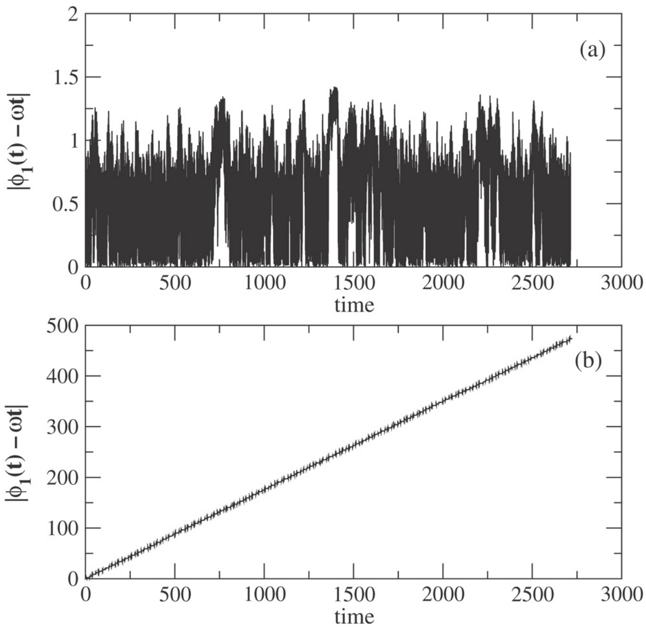


Fig. 1. (a) Phase difference is always smaller than 2π, so PS exists between the circuit and the forcing for $f = 0.2801$ and $V = 0.0015$. (b) PS is not present and the phase difference grows bigger than 2π.

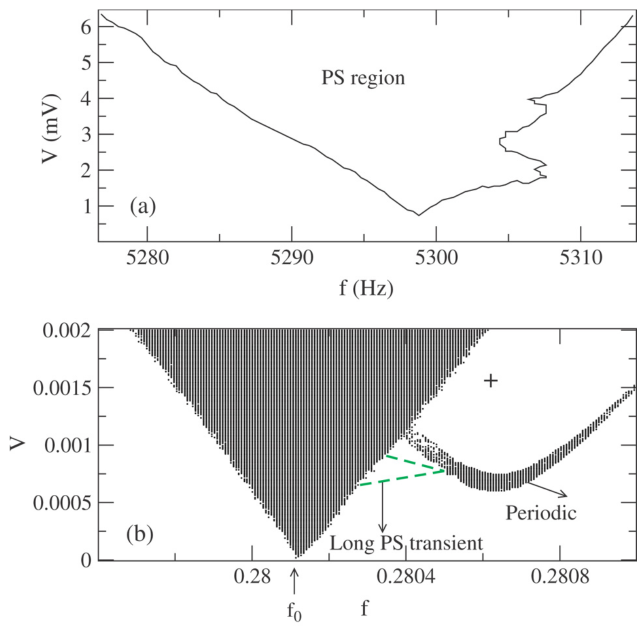


Fig. 2. (a) Experimental PS parameter space. (b) Simulated PS parameter space. Black points represent parameters for which Eq. (6) is satisfied for 120,000 crossings of the trajectory at $X_2 = 0$. In both figures, the horizontal axis represents the forcing frequency f and the vertical axis its amplitude V. Variables in (b) are dimensionless and f_0 is the main frequency of the non-forced circuit.

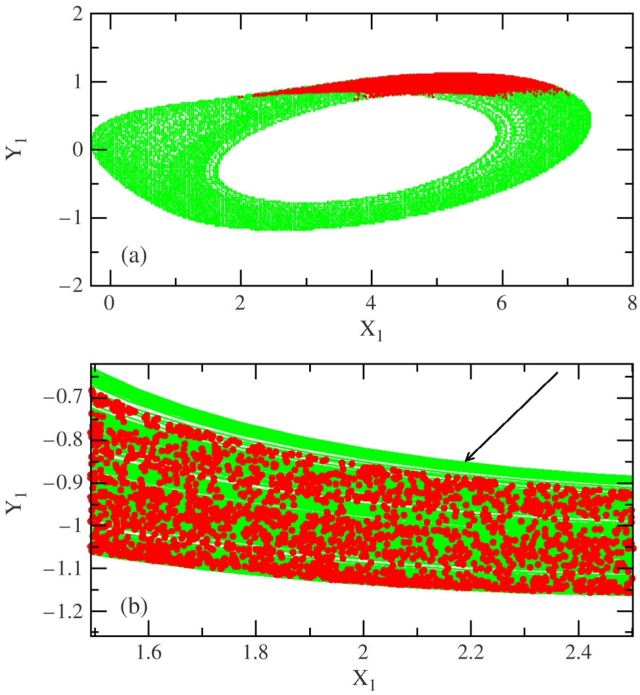


Fig. 3. A projection of the attractor in light gray, and its stroboscopic map in dark gray. The parameters are $f = 0.2801$ and $V = 0.0015$ in (a) and $f = 0.28063$ and $V = 0.0016$ in (b).

for which the stroboscopic map has points concentrated in an angular section smaller than 2π. The bump at the bottom right-hand side of the PS region is due to non-synchronous states that present a long bounded phase difference. In a typical time interval within which the experiment is realized, of the order of 40,000 cycles, the system seemed to be phase synchronized, i.e. localized stroboscopic maps were found. In fact, we have detected these maps even for observation times corresponding to 150,000 cycles, in the region of the bump.

In short, the bump region is an extended structure in the circuit parameter space that presents intermittent behavior in the phase difference [24], but with a long laminar regime, even for parameters far from the border between the PS and the non-PS region. This intermittency differs from the usual one, observed in the transition to PS, in that the latter occurs very close to the border between the PS and the non-PS region. The reason for this intermittency is due to the presence of a periodic window close to the region of the bump.

A simulation is shown in Fig. 2(b), where black points represent perturbing parameters for which Eq. (6) is satisfied, with ϕ_1 defined in Eq. (1). One sees that the PS region resembles a triangle. The triangular shaped region, denoted by the light gray dashed line, represents the region where the system is not phase synchronized, but the phase difference remains bounded for a long time interval, which might be longer than 100,000 cycles of the systems. So, we reproduce numerically the same atypical intermittency observed experimentally, i.e. a long laminar regime in the phase difference, for parameter regions away from the border between PS and non-PS states. This is associated with a periodic window, such as that shown in Fig. 2(b).

The shape of the synchronization region in Fig. 2(b) is equivalent to the region in the experiment, constructed by detecting the stroboscopic maps contained within a small angular section. This proposes an equivalence between the existence of a recurrent subset and the verification of Eq. (6). Inside the synchronization region, a stroboscopic map appears as in Fig. 3(a), where the light gray points represent the attractor, and the dark filled circles the map.

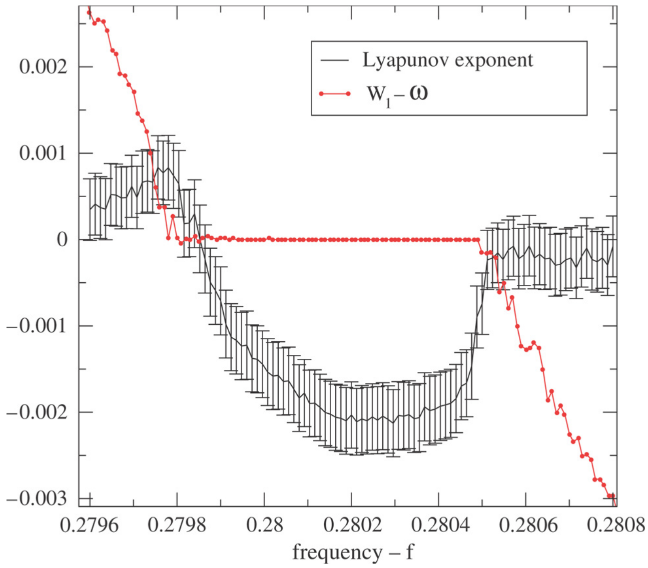


Fig. 4. The Lyapunov exponent (and the error bars) associated with PS in black, and the quantity $\langle W_1 \rangle - \omega$ in gray, for a fixed amplitude of $V = 0.0015$ and a varying frequency.

Outside the PS region there are parameter sets for which the stroboscopic maps do not occupy the region occupied by the attractor at the projection in which the phase is calculated. As one sees in Fig. 3(b) (for the parameters represented by the plus symbol in Fig. 2(b)), there is a region of the projected attractor (indicated by the arrow) which the stroboscopic map never visits.

The difference between the stroboscopic map that appears while there is PS in (a) and the stroboscopic map that appears while there is not PS in (b) is that points of the stroboscopic map in (a) are all concentrated in an angular section of the attractor. As will be classified further, the stroboscopic map in (a) is a PS-set and the stroboscopic map in (b) is a P-set.

As a way to better characterize the PS phenomena in the Chua circuit, we calculate the Lyapunov exponents. As is usually expected, the transition to PS is associated with one of the exponents becoming smaller than zero. Since there is already one exponent that is smaller than zero, PS induces the creation of a second stable direction in the Chua circuit. Prior to the transition to phase synchronization, this exponent was zero. In Fig. 4, we show the exponent (and the error bars) associated with PS in black and the quantity $\langle W_1 \rangle - \omega$ in gray, for a fixed amplitude of $V = 0.0015$, with respect to the frequency. In the region that the exponent becomes smaller than zero, $\langle W_1 \rangle = \omega$, satisfying Eq. (7).

5. Conditional Poincaré map

The finding of maps of the attractor that appear as localized structures implies PS. The conditional Poincaré map introduced in this chapter as a generalization of the stroboscopic map is an efficient way of revealing the existence of such special mappings.

The stroboscopic map technique defined for periodically driven chaotic systems was explained in the previous sections. To extend the idea of the stroboscopic map in coupled chaotic oscillators, we came up with the conditional Poincaré map, which is a map of the flow constructed by observing it for specific times at which events occur in the subsystems $\mathcal{X}_j$. An event is considered to occur when the trajectory of one subsystem crosses a Poincaré section. So, given two oscillators S_1 and S_2 (at least one being chaotic) with trajectories in the subspaces where the phase is defined, the conditional Poincaré map of S_1 is the trajectory position at the moment that a series of equal events happens in S_2. Analogously, the conditional Poincaré map of S_2 is the trajectory position at the moment that a series of equal events happens in S_1. In the case of periodically forced chaotic systems, an event may be defined as happening whenever the forcing reaches a specific value, and the conditional Poincaré map is the usual stroboscopic map, because the time interval between two events is constant. In coupled chaotic systems, the time interval between two successive events is no longer constant.

We define a time series of events τ_j^i by the following rule:

- τ_1^i represents the time at which the i-th crossing of the trajectory of S_2 occurs in a Poincaré plane.
- τ_2^i represents the time at which the i-th crossing of the trajectory of S_1 occurs in a Poincaré plane.

The discrete set of points observed at times τ_j^i is called set $\mathcal{D}$. This set, projected at the subspaces $\mathcal{P}_j$ (where the phase is calculated), is named $\mathcal{D}_j$. The conditional Poincaré map is represented by $T^{\tau_j^i}$. So, we say that $\mathcal{D}_j$ is the set of points generated by $T^{\tau_j^i}$.

The next step is to define when $\mathcal{D}_j$ can be regarded as either a P-set or a PS-set, this last set implying phase synchronization.

6. Sets generated by the conditional Poincaré map

The $\mathcal{D}_j$ set is a P-set if it does not completely fulfill the projection $\mathcal{X}_j$ of the attractor. In other words, a discrete set $\mathcal{D}_j$ is considered to be a P-set if, for balls of radius δ centered in all points of the attractor projection $\mathcal{X}_j$, one does not find points of $\mathcal{D}_j$ inside all these balls. If $\mathcal{D}_j$ completely fulfill $\mathcal{X}_j$, we say that these two sets are equivalent (and we represent this by the symbol $\equiv$). For more details, see Appendix A.

So, a P-set exists if the conditional Poincaré map is not transitive on $\mathcal{X}_j$ [18]. That is, the flow, observed by the times for which the conditional Poincaré map is defined, does not visit arbitrary regions of $\mathcal{X}_j$. Note that, however, the attractor is chaotic, and therefore the chaotic set is always transitive through the flow. So, given a set of initial conditions, its evolution through the flow eventually reaches arbitrary open subsets of the original chaotic attractor.

We can classify three relevant types of sets generated by the conditional Poincaré map:

Type-a $\mathcal{D}_j$ is equivalent to $\mathcal{X}_j$ ($\mathcal{D}_j \equiv \mathcal{X}_j$). The conditional Poincaré map $T^{\tau_j^i}$ is transitive in $\mathcal{X}_j$.
P-set $\mathcal{D}_j$ is NOT equivalent to $\mathcal{X}_j$ ($\mathcal{D}_j \not\equiv \mathcal{X}_j$). The conditional Poincaré map is NOT transitive in $\mathcal{X}_j$.
PS-set $\mathcal{D}_j$ is P-set, with the additional condition that it is localized in the vicinity of the Poincaré section chosen to define the events.

In the following, we comment on each case.

6.1. *Type-a sets*

6.1.1. *These sets appear whenever there is no PS*

If two non-identical coupled chaotic systems (topologically similar) are not phase synchronized, the chaotic trajectories do not make correlated events in both subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$. As a consequence, while the trajectory is positioned at the specified Poincaré plane at the subspace $\mathcal{P}_1$, the trajectory in the subspace $\mathcal{P}_2$ is everywhere in this subspace, making the set $\mathcal{D}_j$ equivalent to $\mathcal{X}_j$.

An interesting illustration is the case of two uncoupled equal chaotic systems, but with different initial conditions. As we construct the conditional Poincaré maps, they will be a type-a set, since the distance between the trajectories in the two oscillators is also sensitive to the initial conditions, and will diverge exponentially.

6.2. P-sets

6.2.1. These sets constitute a necessary, but not sufficient, condition to describe PS

In periodically forced chaotic systems, these might appear when there is no PS. As we have already mentioned, the points in these sets are not localized in special spots of the attractor projection. As a consequence, the domain of the absolute difference between the times at which the same number of events happen in both oscillators has a broad character.

6.3. PS-sets

6.3.1. PS-sets imply PS, and vice-versa

These exist, if and only if, there is phase synchronization, as shown in Appendix B and illustrated with the examples in Section 4 and throughout this work. Another important point is that the PS-set provides a real-time detection that can easily be implemented experimentally (Section 4) and constructed from a data set.

PS-set implies PS because the difference between the time at which the N-th event happens in both oscillators is small, which means that the time difference $|\tau_1^N - \tau_2^N|$ is smaller than a finite constant value. As a consequence, the points in the conditional Poincaré map of one oscillator are confined around the Poincaré section chosen to define the events. Therefore, the detection of a PS-set can be performed by observing this characteristic of the conditional Poincaré map.

For Rössler-like oscillators, in which the trajectory spirals around an equilibrium point, the PS-set is confined within an angular section.

6.4. Length of a PS-set

Having found the PS-set, we can study the properties of these sets that give us the level of organization and coherence of the oscillators.

A PS-set, $\mathcal{D}_j$, is said to have length 1 if the set is constructed by the time series of the same events. Defining the event to be given by the crossing of the trajectory to a Poincaré plane, the corresponding time series of events $M_j(1)$ is given by $\tau_j^i, \tau_j^{i+1}, \tau_j^{i+2}, \tau_j^{i+3}, \ldots$. For this PS-set, points in $\mathcal{D}_j$ are mapped in $\mathcal{D}_j$ after one application of the conditional Poincaré map.

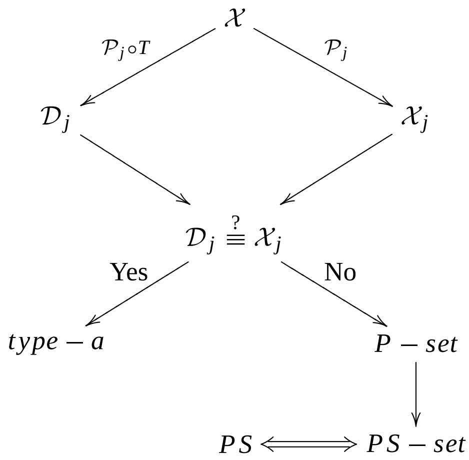
A PS-set is said to have length 2 if it is obtained by a time series of two different events. So, a PS-set can be constructed from a more complex series of events. We construct a length-2 basic set using a time series of events $M_j(2)$ given by $\tau_j^i, \frac{\tau_j^i+\tau_j^{i+1}}{2}, \tau_j^{i+1}, \frac{\tau_j^{i+1}+\tau_j^{i+2}}{2}, \ldots$ [25]. As an example, for the perturbed Chua circuit, the length-2 basic set is constructed by a stroboscopic map that collects points every half-period of the forcing. A length-2 basic set is assumed to be composed by two other subsets, named minimal sets (the subsets $\mathcal{D}_j^0$ and $\mathcal{D}_j^1$).

These have the property that, if a point x_0 is such that $x_0 \in \mathcal{D}_j^0$, this point, iterated by the conditional Poincaré map, goes to the minimal set $\mathcal{D}_j^1$; if x_0 is such that $x_0 \in \mathcal{D}_j^1$, this point, iterated by the conditional Poincaré map, goes to the minimal set $\mathcal{D}_j^1$. Thus, points in $\mathcal{D}_j^1$ are mapped to itself after two applications of the conditional Poincaré map. The minimal set $\mathcal{D}_j^1$ is said to be disjoint to the set $\mathcal{D}_j^2$ if they do not intersect, i.e. $\mathcal{D}_j^1 \cap \mathcal{D}_j^2 = \emptyset$. For some systems that present a strong phase-coherent state, such as those studied here in which the instantaneous trajectory velocity does not differ too much from the average velocity on typical orbits, it is possible to find a length-2 basic set, with disjoint minimal sets, when PS is present.

For a general case, we do not expect to find a length-2 PS-set with disjoint minimal sets. As an example, one can think of a spiking-firing oscillator, phase synchronized with a periodic forcing. Due to the fact that the spiking-firing dynamics have a fast mode and a slow mode, the conditional Poincaré maps might overlap.

6.5. Set diagram

Here we explain, through a diagram, the possible emerging sets from the conditional Poincaré map.



Starting from the chaotic attractor $\mathcal{X}$, the set $\mathcal{D}$ is constructed from the conditional Poincaré map represented by T; we project $\mathcal{D}$ and $\mathcal{X}$ into the subspace $\mathcal{P}_j$, obtaining the sets $\mathcal{D}_j$ and $\mathcal{X}_j$, respectively.

We classify the set $\mathcal{D}_j$ into type-a set or P-set by checking whether the conditional Poincaré map is transitive in $\mathcal{X}_j$, i.e. by verifying the equivalence between the sets $\mathcal{X}_j$ and $\mathcal{D}_j$. Then, if the P-set is localized in the vicinity of the Poincaré section where the events occur, the P-set is a PS-set, which means that PS is present.

7. PS-sets in the Chua circuit

For applying our formalism to the periodically forced Chua circuit, the event times are $\tau_1^i = i\tau$, with τ representing the forcing period. The time series for the length-2 basic set is given by $1/2\tau, \tau, 3/2\tau, 2\tau, 5/2\tau, 3\tau, \ldots$. Everywhere inside the PS region, we find length-1 (as an example, see Fig. 3(a)) and length-2 (as an example, see Fig. 5(a)) PS-sets, the latter with disjoint minimal sets. In Fig. 5(a), the application of the conditional Poincaré map in the minimal set $\mathcal{D}_1^0$ leads to the minimal set $\mathcal{D}_1^1$. Both sets form a length-2 PS-set. Note that both $\mathcal{D}_1^0$ and $\mathcal{D}_1^1$ can be regarded as a length-1 PS-set. To our numerical precision, we have checked that there are no basic sets beneath the synchronization region tip. This means that the type-a set is present for very small but finite amplitude forcing. Outside the PS region, there is a P-set, i.e. a non-transitive conditional Poincaré map on the chaotic attractor projection. So, $\mathcal{D}_j \not\equiv \mathcal{X}_j$. In this case, there is no PS. More examples of length-Q PS-sets in the phase synchronous forced Chua circuit can be seen in [26].

8. PS-sets in the coupled Rössler system

We can use the formalism of the conditional Poincaré map to study the appearance of PS in coupled chaotic systems, as the two coupled Rössler oscillators are given by:

$$\dot{x}_{1,2} = -\alpha_{1,2} y_1 - z_1 + \epsilon(x_{2,1} - x_{1,2}) \tag{12}$$

$$\dot{y}_{1,2} = \alpha_{1,2} x_1 + 0.15 y_1 \tag{13}$$

$$\dot{z}_{1,2} = 0.2 + z_1(x_1 - 10) \tag{14}$$

with $\alpha_1 = 1$ and $\alpha_2 = \alpha_1 + \delta\alpha$. The index denotes systems 1 and 2. The subspaces $\mathcal{P}_j$ are defined by $\mathcal{P}_j = (x_j, y_j)$. In a coupled chaotic system, τ_1^i (τ_2^i) does not increase uniformly, but it is given by the time the trajectory crosses

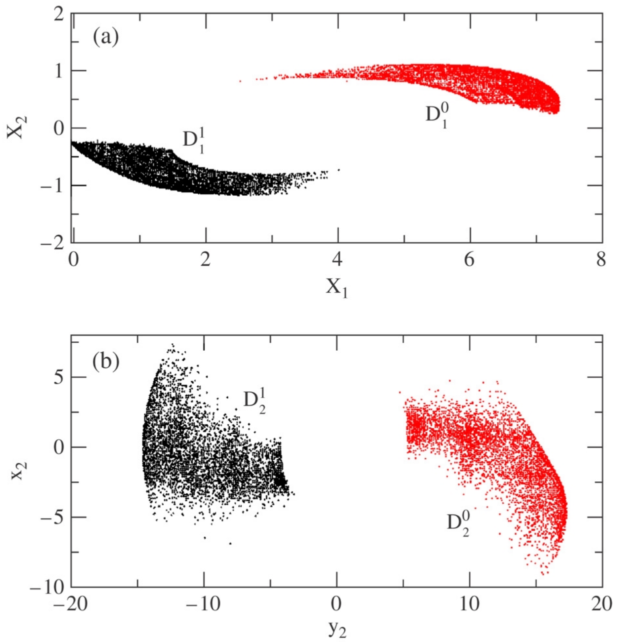


Fig. 5. (a) Length-2 PS-set with disjoint minimal sets, in the forced Chua circuit, for the parameters $f = 0.2801$ and $V = 0.001$. The PS-set is constructed by the time series $M_1(2)$. (b) Length-2 PS-set with disjoint minimal sets, in the coupled Rössler oscillators, for the parameters $\epsilon = 0.01$ and $\delta\alpha = 0.001$. The PS-set is constructed by the time series $M_2(2)$.

the Poincaré plane $y_2 = 0$ ($y_1 = 0$). For these times, and using the time series $M_j(2)$, we construct the minimal sets $\mathcal{D}_1^0$ and $\mathcal{D}_1^1$ ($\mathcal{D}_2^0$ and $\mathcal{D}_2^1$). The parameters are $\epsilon = 0.01$ and $\delta\alpha = 0.001$.

In Fig. 5(b), we show a length-2 basic set with disjoint minimal sets. The application of the conditional Poincaré map in the minimal set $\mathcal{D}_2^0$ leads to the minimal set $\mathcal{D}_2^1$, and vice-versa. These two sets together form a length-2 PS-set, but each one separately can be regarded as a length-1 PS-set. The characteristic of these PS-sets is that they appear as localized structures around the Poincaré section chosen to define the events. As one can see, the set $\mathcal{D}_2^0$ is localized in the neighborhood of the line $y_2 = 0$, where the Poincaré section is chosen.

In fact, this length-2 PS-set with disjoint minimal sets (as well as a length-1 PS-set) is found everywhere in the PS region, as shown in Fig. 6. In it, filled squares represent parameters for which these special PS-sets are found, and empty circles parameters for which PS exists.

A PS-set of length Q is detected using $\mathcal{D}_2^{Q-1}$ in Eq. (A.2), from which we can check whether $\mathcal{D}_2^Q$ occupies (type-a discrete set $\mathcal{D}$) the whole space occupied by $\mathcal{X}_1$. The set $B_\ell(x)$ in Eq. (A.2) is constructed assuming squares of size $\ell = 1.5$ in points of the set $\mathcal{D}_2^{Q-1}$. In Fig. 6, we show a case for $Q = 2$.

In Fig. 7(a), we show the attractor in the subspace $\mathcal{P}_1$, the subset $\mathcal{X}_1$ in gray, and the discrete set $\mathcal{D}_1$ in dark empty circles. In (b), we show a magnification of the box in (a). Note that neighborhoods of arbitrary points in the trajectory of $\mathcal{X}_1$ (gray) always contain a point of the discrete set $\mathcal{D}_1$ (dark empty circles). So, the set $\mathcal{X}_1$ is equivalent to the set $\mathcal{D}_1$ and, therefore, $\mathcal{D}_1$ is not a PS-set, and therefore it does not exist PS. In contrast to the Chua circuit, the Rössler coupled system presents no P-sets for parameters outside the PS region.

9. Extension to other coupled chaotic systems

The approach presented in this paper can be extended for non-coherent attractors. As noticed in Ref. [20], attractors that present non-coherent phase motion in the phase space may present a coherent motion in the space of

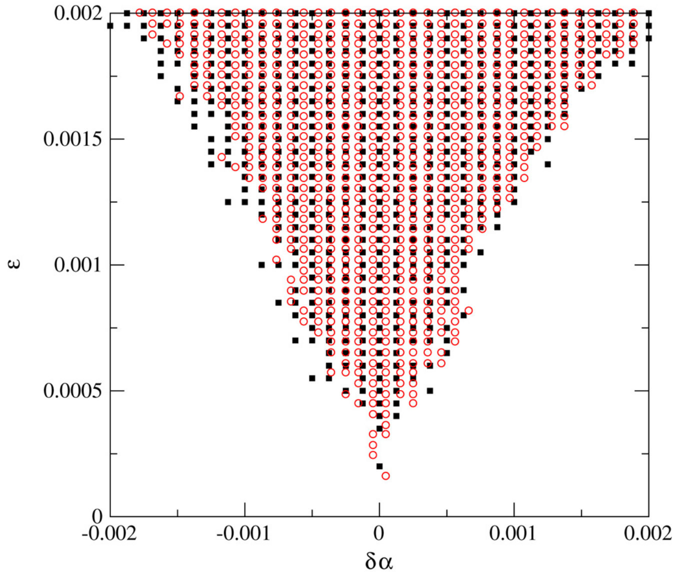


Fig. 6. Empty circles show parameters for which PS exists, detected by Eq. (6), and filled squares represent parameters for which a length-1 PS-set appears simultaneously with a length-2 PS-set with disjoint minimal sets. The horizontal axis represents parameter mismatch and the vertical axis the coupling amplitude.

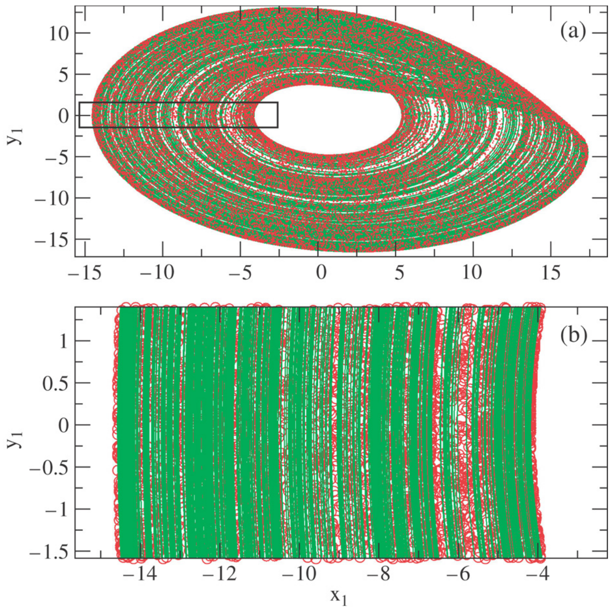


Fig. 7. We show a situation where PS is not found for $\epsilon = 0.01$ and $\delta\alpha = 0.00001$. In (a), we show the attractor in the subspace $\mathcal{P}_1$, the subset $\mathcal{X}_1$ in gray, and the discrete set $\mathcal{D}_1$ in dark empty circles. In this figure, $\mathcal{D}_1$ is a type-a set, since $\mathcal{X}_1$ visits every neighborhood of points in $\mathcal{D}_1$ (the conditional Poincaré map is transitive in $\mathcal{X}_1$). In (b), we show a magnification of the box in (a).

the velocities, i.e. $(\dot{x}, \dot{y})$, which is the case of the funnel attractor [20]. In this case, the extension of our approach to the non-coherent phase motion is straightforward. Instead of defining a conditional Poincaré map in the phase space, we analyze the dynamics in the velocity space, in which the phase is coherent, and therefore all the ideas introduced herein can be applied in this new space.

For some chaotic attractors, it might not be possible to define a Poincaré section to construct the conditional Poincaré map. However, one can still construct these conditional maps by defining different types of events. For example, in coupled neurons, this event can be chosen as the beginning or end of the bursts, which can be defined well by the crossing of the trajectory with some given threshold.

10. Conclusion and remarks

A chaotic set is always transitive through the flow. So, given a set of initial conditions, its evolution through the flow eventually reaches arbitrary open subsets of the original chaotic attractor. However, a stroboscopic map of the flow, whose generalization here is called a conditional Poincaré map, might not possess the transitive property. That is, given a set of initial conditions, its evolution through the conditional Poincaré map *might* not reach arbitrary open subsets of the chaotic attractor.

The introduction of the term "conditional" in the map nomenclature comes from the unconventional and non-rigid way we adapt the established definition of a stroboscopic Poincaré map. For coupled chaotic oscillators, this conditional map is constructed on the basis of events that are conveniently chosen at the same subspaces where the phase of a chaotic system is defined. In addition, the application of this map through the flow, which results in a discrete set, is inspected not in the whole phase space but in the same subspaces where the phase is defined.

If phase synchronization exists, the conditional map generates special discrete sets, named PS-sets. The contrary is also true, i.e. a PS-set implies phase synchronization. This was illustrated in the periodically forced Chua circuit, in the coupled Rössler oscillator, and in other more general (topologically equivalent) coupled chaotic systems.

The ideas introduced here provide an efficient way of detecting phase synchronization without having to actually measure the phase of the chaotic trajectory. Indeed, this detection can be performed in experiments in real time, as was done here in the perturbed Chua circuit.

It is worth saying that the PS-sets are robust under small additive noise that could corrupt the data in an experiment. This is so because a small additive noise does not interfere much with the time that the trajectory crosses the Poincaré section, but just deviates the timing of the crossing. Then, the PS-sets remains. This robustness against the noise is an important property for applying these ideas in experiments, as is done in this work.

We have also introduced the phase as a quantity that measures the velocity of rotation of a projection of the tangent vector along the trajectory. This definition can be applied to arbitrary flows, independent of whether or not they present coherent or non-coherent phase dynamics.

Finally, our formalism for the conditional Poincaré map can be used in coupled maps (or perturbed) to detect synchronous behavior (not **phase** synchronization) between the systems (or between the forcings) for the case where one does not find full synchronization between the maps. One particular example where this happens is in the periodically forced logistic equation [27] or for a system of coupled logistic maps [28], where one can find a finite number of synchronous chaotic subsets (the basic sets).

Acknowledgment

Research partially financed by FAPESP, Alexander Von Humboldt Foundation, CNPq, Helmholtz Center for Mind and Brain Dynamics.

Appendix A. The conditional Poincaré map and the $\mathcal{D}_j$ sets

Next, we present some formalism using elements of Ergodic Theory in order to introduce the conditional Poincaré map and the $\mathcal{D}_j$ sets.

Given a flow F_t, we call $\mathcal{X}$ the chaotic attractor. A subset of the attractor is its projection on the subspace $\mathcal{P}_j$, which we call $\mathcal{X}_j$. We assume $\mathcal{X} \in R^d$.

The notion of the stroboscopic map of a flow can be generalized to any temporal translation on the trajectory. We define the temporal translation to be a transformation represented by $T^{\tau_j^i}$, called the conditional Poincaré map. The initial condition $\vec{x}(t)$ is iterated under the temporal transformation $T^{\tau_j^i}$ to the point $\vec{x}(t + \tau_j^i)$. Applying the transformation $T^{\tau_j^i}$ in a typical trajectory for the time sequence τ_j^i, τ_j^{i+1}, τ_j^{i+2} gives us the points $\vec{x}(\tau_j^i)$, $\vec{x}(\tau_j^{i+1})$, and $\vec{x}(\tau_j^{i+2})$. So, $\vec{x}(\tau_j^{i+1})$ is the point $\vec{x}(\tau_j^i)$ integrated by the flow for a time interval given by $\tau_j^{i+1} - \tau_j^i$. Applying the transformation $T^{\tau_j^i}$ for an infinite series of τ_j^i, that is, $i = 1, 2, 3, \ldots, \infty$, gives us a discrete set that we call $\mathcal{D}$. The projection of $\mathcal{D}$ on the subspace $\mathcal{P}_j$ is named $\mathcal{D}_j$.

Now, we introduce the notion of transitivity. Let us assume that we have a chaotic set $\mathcal{A}$. Let us choose two disjoint subsets $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{A}$. So, $\mathcal{B} \cap \mathcal{C} = \emptyset$. There is a transformation F that generates the set $\mathcal{A}$, with the property that $T(\mathcal{B}) = \mathcal{C}$ and $F(\mathcal{C}) = \mathcal{B}$. So, clearly the transformation F and its n-fold application (F^n) can always place an arbitrary initial condition, belonging either in $\mathcal{B}$ or $\mathcal{C}$, anywhere in the set $\mathcal{A}$. This property makes the transformation F transitive in the set $\mathcal{A}$. However, the conditional Poincaré map might no longer be transitive.

For example, in the case of the $2n$-fold of the transformation T, i.e. T^{2n}, given an initial condition in the set $\mathcal{B}$, its iteration by T^{2n} will never reach the set $\mathcal{C}$. Therefore, the conditional Poincaré map T^{2n} is not a transitive transformation in the set $\mathcal{A}$. In the case of flows, we have seen that the temporal transformation represented by $T^{\tau_j^i}$, whenever PS is present, does not place arbitrary points of the attractor everywhere in the subsets $\mathcal{X}_j$ of the chaotic attractor. Therefore, if PS is present, the transformation $T^{\tau_j^i}$ is not transitive to this subset of the attractor.

The conditional Poincaré map $T^{\tau_j^i}$ is topologically transitive in a set $\mathcal{A}$ [29] if, for any two open sets $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$,

$$\exists \tau_j^i / T^{\tau_j^i}(B) \cap C \neq \emptyset. \tag{A.1}$$

To detect whether the conditional Poincaré map is transitive in some subset, we introduce the notion of equivalence. Two sets $\mathcal{A}$ and $\mathcal{B}$ are (not) equivalent (where equivalence is represented by the symbol $\equiv$, and non-equivalence by $\not\equiv$) if they (do not) occupy the same neighboring space. In a more general way:

Definition 1. Two sets $\mathcal{A}$ and $\mathcal{B}$ are equivalent, $\mathcal{A} \equiv \mathcal{B}$, if $\forall x \in \mathcal{A}$, a set $\mathcal{C}$ can be constructed by the union of open sets $B_\ell(x)$, open R^d volumes centered at x with length ℓ, such that $\forall y \in \mathcal{B} \Longrightarrow y \in \mathcal{C}$, and $\mathcal{A} \not\equiv \mathcal{B}$ if $y \notin \mathcal{C}$.

Definition 2. The set $\mathcal{D}$ is a P-set if $\mathcal{X}_j \not\equiv \mathcal{D}_j$.

Definition 3. If $\mathcal{D}_j$ can be decomposed into a collection $\mathcal{D}_j = \mathcal{D}_j^0, \mathcal{D}_j^1, \ldots \mathcal{D}_j^{Q-1}$ of subsets of $\mathcal{X}_j$, with $Q \geq 1$, such that a point in $\mathcal{D}_j^i$ iterated by the conditional Poincaré map goes to $\mathcal{D}_j^{i+1(\text{mod } Q)}$, we refer to each minimal set $\mathcal{D}_j^i$ as a *recurrent decomposition*. In the particular case where we have $\tau_j^{i+1} - \tau_j^i = \tau$ (constant), this minimal set is a *periodic decomposition*. The number of sets Q is called the *length* of the decomposition [30].

Proposition 1. *If $T^{\tau_j^i}$ is transitive in $\mathcal{X}_j$, then $\mathcal{D}_j \equiv \mathcal{X}_j$.*

Proposition 2. *If $T^{\tau_j^i}$ is non-transitive in $\mathcal{X}_j$, then a subset $\mathcal{A}$ can be constructed such that $\mathcal{A} \subset \mathcal{X}_j$ and $\mathcal{A} \not\equiv \mathcal{X}_j$.*

Proofs of Propositions 1 and 2 can be performed by using Eq. (A.1) and the definitions.

Definition 3 can be understood from the PS-set of length-1 and length-2 considered in this work. If the length-1 PS-set is constructed by the time series $\tau_j^i, \tau_j^{i+1}, \tau_j^{i+2}, \tau_j^{i+3}, \ldots$, the length-2 PS-set is obtained from the time series given by $\tau_j^i, \frac{\tau_j^i + \tau_j^{i+1}}{2}, \tau_j^{i+1}, \frac{\tau_j^{i+1} + \tau_j^{i+2}}{2}, \ldots$ [25]. For the particular case of a periodically forced system,

$\tau_1^{i+1} - \tau_1^i = 1/f$, so the length-1 PS-set is the standard stroboscopic map that collects points every period of the forcing. The length-2 PS-set is constructed by a stroboscopic map that collects points every half-period of the forcing. For the length-2 PS-set, we have two minimal sets, the sets $\mathcal{D}_j^0$ and $\mathcal{D}_j^1$, with the property that, if $x_0 \in \mathcal{D}_j^0$, this point iterated by the conditional Poincaré map goes to $\mathcal{D}_j^1$ and $\mathcal{D}_j^1$ goes to $\mathcal{D}_j^0$, under the conditional Poincaré map.

To check whether $\mathcal{D}_j \not\equiv \mathcal{X}_j$, we perform the following: for $x \in \mathcal{X}_j$ there exists $y \in \mathcal{D}_j$ such that

$$y \cap B_\ell(x) = \emptyset, \tag{A.2}$$

where $B_\ell(x)$ is an open ball of radius δ centered at the point x. δ is a small positive value.

Appendix B. The PS-sets

In this appendix, we show that a PS-set exists if, and only if, phase synchronization exists; in other words, PS-set implies PS and PS implies PS-set.

Given a dynamical system $\mathbf{Y}' = G(\mathbf{Y})$, and let F^t be the flow and $\mathcal{X}$ the attractor generated by it, we suppose that we have chaotic dynamics from now on. Let Σ_j be the Poincaré section in the subspace $\mathcal{P}_j$, and let Π_j be the Poincaré map associated with the section Σ_j, such that, given a point $x_j^i \in \Sigma_j$, $x_j^{i+1} = \Pi_j(x_j^i) = F^{\Delta\tau_j^{i+1}}(x_j^i)$. From now on, we use a rescaled time $t' = t/\langle T_1 \rangle$. For a slight abuse of notation, we omit the symbol $'$.

Proposition 3. *Given two interacting oscillators, PS-sets can be constructed if, and only if, phase synchronization is present.*

To show the *if* in the preposition, let us start by considering that the time interval associated with the return of the point x_j^i to the point x_j^{i+1} is $\Delta\tau_j^{i+1} = \tau_j^{i+1} - \tau_j^i$, with τ_j^{i+1} being the times at which the subsystem $\mathcal{X}_j$ crosses the Poincaré section Σ_j in the subspace $\mathcal{P}_j$.

As has already been introduced, the average return time is given by $\langle T_j \rangle = \frac{\sum_{i=0}^{N} \Delta\tau_1^j}{N} = \frac{\tau_j^N}{N}$, and the time is rescaled such that $\langle T_1 \rangle = 1$. Our hypothesis is that the subsystem $\mathcal{X}_j$ has a phase-coherent oscillation, so there is a number δ_j for which the following holds [31]:

$$|\tau_j^N - N| \le \delta_j. \tag{B.1}$$

The number $\delta_j < 1$ measures the coherence in the phase oscillation, and is linked to the phase diffusion [7,31]. This equation holds for all N, so it implies that, for a single oscillation, it is also true that $|\Delta\tau_j^i - \langle T_j \rangle| \le \delta_j$. In the case of two systems that present PS, the following holds [21]:

$$|\tau_1^N - \tau_2^N| \le \delta_3, \tag{B.2}$$

with $\delta_3 < 1$. This equation implies that $|\Delta\tau_1^i - \Delta\tau_2^i| \le \delta_3$, which states that the time intervals in a single oscillation are strongly related in phase synchronization.

Let us introduce a new variable that measures the difference between the time interval of two events in $\mathcal{X}_1$ and $\mathcal{X}_2$. This new variable is $\Delta\tau_{2,1}^i = \Delta\tau_2^i - \Delta\tau_1^i$. From Eq. (B.2), it is true that $|\Delta\tau_{2,1}^i| \le \delta_3$.

Now, we analyze one typical oscillation. Given the following initial conditions $x_1^0 \in \Sigma_1$ and $x_2^0 \in \Sigma_2$, we evolve both until x_1^0 returns to Σ_1. In other words, we evolve both initial conditions for a time $\Delta\tau_2^1$. So, $F^{\Delta\tau_2^1}(x_1^0) = \Pi_2(x_1^0) = x_1^1 \in \Sigma_1$. Analogously, $F^{\Delta\tau_2^1}(x_2^0) = F^{\Delta\tau_1^1 + \Delta\tau_{2,1}^1}(x_2^0) = F^{\Delta\tau_{2,1}^1} \circ F^{\Delta\tau_1^1}(x_2^0) = F^{\Delta\tau_{2,1}^1}(x_2^1)$.

Now, we use the fact that $|\Delta\tau_{2,1}^i| < \delta_3$, and write that

$$F^{\Delta\tau_{2,1}^1}(x_2^1) \approx x_2^1 + \mathbf{G}(x_2^1)\delta_3. \tag{B.3}$$

So, given an initial condition in Σ_1 (subsystem $\mathcal{X}_1$) evaluated by the time interval $\Delta\tau_2^i$ (of subsystem $\mathcal{X}_2$), it returns near the section Σ_1.

For a general case, we have to show that an initial condition, on the section Σ_1, evolved by the flow for the time $\sum_{i=0}^{N} \Delta\tau_{2,1}^i$ still remains close to this section. In other words, we have to show that the approximation in Eq. (B.3) is valid for an arbitrary number of events N in the subspace $\mathcal{X}_2$. Now, noting that $\sum_{i=0}^{N} \Delta\tau_{2,1}^i = \tau_2^N - \tau_1^N$, from our hypotheses of phase-coherent dynamics $|\sum_{i=0}^{N} \Delta\tau_{2,1}^N| = |\tau_2^N - \tau_1^N| < \delta_3$. The same arguments that were used to derive Eq. (B.3) for one oscillation can be extended to an arbitrary number N, so we prove the *if* of the proposition (PS implies the existence of PS-sets).

Now, to show the *only if* (PS-sets imply PS), let us say that there is a PS-set. As a consequence, Eq. (B.2) is valid. Then, using the definition of phase coherence [31] and noting that in PS the average return times in a given Poincaré section are the same, we see that Eq. (B.2) comes from the boundedness of the phase, concluding the proposition.

Furthermore, let us suppose that the trajectories of the oscillators are perturbed by a small perturbation that does not destroy the phase synchronous dynamics. The effect of the small perturbation in the return time of the trajectory to a Poincaré section is to deviate this time according to $\bar{T}_j^i = T_j^i + \xi_j^i$, where ξ_j^i represents the perturbation in system S_j at the moment of the i-th event, with $\max_i |\xi_j^i| < \kappa$. Under these hypotheses about the perturbation, we conclude the following result.

Proposition 4. *The PS-set is robust under perturbations.*

This result shows that a PS-set can be constructed in PS states. Moreover, for the coupled Rössler-like systems, this result states that this set is confined in an angular region, which is a consequence of Eq. (B.2).

To see the relationship between the constant δ_3 and the size of the PS-set, we perform the following. From the time normalization, the average time interval between points in phase space is proportional to the distances between them. So, from Eq. (B.3), we write $\delta_3 = |\mathcal{H}|/\mathbf{G}(x_2^0)$, with $\mathcal{H}$ being the average half-length of the PS-set. A rough calculation shows that, in our experiment with the Chua circuit, we have that $\delta_3 \approx 1/2.5$, and for the coupled Rössler oscillators we have that $\delta_3 \approx 1/2.1$. These results agree completely with the theoretical approach in [21].

References

[1] B. Blasius, L. Stone, Nature 406 (2000) 846; Nature 399 (1999) 354.
[2] R.D. Pinto, P. Varona, A.R. Volkovskii, A. Szucs, H.D.I. Abarbanel, M.I. Rabinovich, Phys. Rev. E 62 (2000) 2644.
[3] M.V. Ivanchenko, G.V. Osipov, V.D. Shalfeev, J. Kurths, Phys. Rev. Lett. 93 (2004) 134101.
[4] P. De Grauwe, H. wachter, M. Embrechts, Exchange Rate Theory: Chaotic Models of Foreign Exchange Markets, Blackwell, Oxford, 1993.
[5] I. Leyva, E. Allaria, S. Boccaletti et al., Phys. Rev. E 68 (2003) 066209.
[6] I. Fischer, Y. Liu, P. Davis, Phys. Rev. A 62 (2000) 011801.
[7] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, 2001; S. Boccaletti, J. Kurths, G. Osipov, D. Valladares, C. Zhou, Phys. Rep. 366 (2002) 1.
[8] S. Strogatz, Sync: The Emerging Science of Spontaneous Order, Hyperion, New York, 2003.
[9] M.G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. 76 (1996) 1804.
[10] U. Parlitz, L. Junge, W. Lauterborn, L. Kocarev, Experimental observation of phase synchronization, Phys. Rev. E 54 (1996) 2115.
[11] I.Z. Kiss, J.L. Hudson, Phys. Rev. E 64 (2001) 046215.
[12] C.M. Ticos, E. Rosa Jr., W.B. Pardo et al., Phys. Rev. Lett. 85 (2000) 2929.
[13] M.S. Baptista, T.P. Silva, J.C. Sartorelli et al., Phys. Rev. E 67 (2003) 056212.
[14] J. Fell, P. Klaver, C.E. Elger, G. Fernandez, Rev. Neurosci. 13 (2002) 299.
[15] F. Mormann, T. Kreuz, R.G. Andrzejak, P. David, K. Lehnertz, C.E. Elger, Epilepsy Res. 53 (2003) 173.
[16] Phase can be defined as a Hilbert transformation of a trajectory component, as the angle of a trajectory point into a special projection of the attractor, as a function that grows by 2π every time the chaotic trajectory crosses some specific surface, or as the angle of the projected vector field into some subspace with respect to some rotation point.

[17] A. Pikovsky, M. Rosenblum, G.V. Osipov, J. Kürths, Physica D 104 (1997) 219.

[18] A chaotic set is always transitive through the flow. So, given a set of initial conditions, its evolution through the flow eventually reaches arbitrary open subsets of the original chaotic attractor. However, the conditional Poincaré map might not possess the transitive property. That is, given a set of initial conditions, its evolution through this map *might* not reach arbitrary open subsets of the special projections of the chaotic attractor; its dynamics stay confined to a subset of the attractor.

[19] The results presented here for the sinusoidally forced Chua circuit were also verified in the sinusoidally forced Rössler oscillator.

[20] V.G. Osipov, H. Bambi, C. Zhou, V.M. Ivanchenko, J. Kurths, Phys. Rev. Lett. 91 (2003) 024101.

[21] M.S. Baptista, T. Pereira, J. Kurths, Upper bounds in phase synchronous weak coherent chaotic attractors (submitted for publication).

[22] r is considered to be rational. However, as shown in Ref. [23], PS, as defined by the boundedness of the phase difference, was found in two chaotic systems for a finite but very large time interval as r approaches an irrational. Therefore, although in this work we consider r to be rational, we should make the remark that, for the special situation such as that presented in Ref. [23], Eq. (6) can only be satisfied for a finite but large time if r is considered to be irrational.

[23] M.S. Baptista, S. Boccaletti, K. Josić, I. Leyva, Phys. Rev. E 69 (2004) 056228.

[24] The intermittency observed in the phase difference is characterized as a usual intermittency by the alternation between a laminar regime and a burst regime. If the phase difference remains bounded in the interval $[0, \langle\Delta\phi_1\rangle]$, we say that we have a laminar regime. If the phase difference leaves this interval, we have a burst, also known as phase slip. As one approaches the border between the PS and the non-PS region, the laminar regime in the phase difference becomes longer.

[25] The choice of these time intervals is not unique. It depends on what type of event one wants to identify in the system. A length-1 basic set is constructed by measuring time intervals between the occurrence of two events of the same type. A length-2 basic set is constructed by measuring the time interval between the occurrence of an event A and the occurrence of an event B, and then between the event B and finally the event A, and so on.

[26] M.S. Baptista, T. Pereira, J.C. Sartorelli, I.L. Caldas, J. Kurths, 742 AIP Proceedings of the 8-th Experimental Chaos Conference, American Institute of Physics, 2004.

[27] M.S. Baptista, I.L. Caldas, Chaos Solitons Fractals 7 (1996) 325;
M.S. Baptista, I.L. Caldas, Internat. J. Bifur. Chaos 7 (1997) 447.

[28] A. Shabunin, V. Demidov, V. Astakhov et al., Phys. Rev. E 65 (2002) 056215.

[29] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, 1996.

[30] J. Banks, J. Ergod, Th. Dynam. Sys. 17 (1997) 505.

[31] K. Josić, M. Beck, Chaos 13 (2003) 247;
K. Josić, D.J. Mar, Phys. Rev. E 64 (2001) 056234.