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GENERAL AND APPLIED PHYSICS

# Magnetohydrostatic Equilibrium with External Gravitational Fields in Symmetric Systems

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**Abstract** We present a general formulation for magnetohydrostatic equilibria with external gravitational fields in symmetric systems with one ignorable coordinate, using non-orthogonal coordinate systems. We consider the cases of isothermal as well as adiabatic processes. Analytical exact solutions for the ideal magnetohydrodynamical equilibrium equation are presented for rectangular, cylindrical, and spherical coordinates.

**Keywords** Magnetohydrodynamics · MHD equilibrium · Magnetohydrostatics

## 1 Introduction

There are various application of ideal magnetohydrodynamics (MHD) to astrophysical phenomena, in particular to the description of many structures of the solar/stellar atmosphere, where the electric conductivity of the plasma is very large and the viscosity is low enough [1, 2]. In such cases, beside the usual hydrodynamical and Maxwell equations, one has to add an equation for the gravitational field [3]. The general problem of coupling hydromagnetic and gravitational equations, however, is difficult to solve, even numerically.

A useful simplification arises, however, when the gravitational field is external, i.e., does not depend on the plasma itself. This is an acceptable approximation if the plasma density is very small, and the influence of its own gravitational field is negligible when compared with the external field. The hydromagnetic equilibrium of plasmas in a uniform gravitational field has been considered in the classical work of Kippenhahn and Schlüter of 1957, describing quiescent solar prominences in rectangular coordinates [4].

Analytical solutions for rectangular coordinates have been studied by Khater et al., by using Painlevé analysis method [5]. Other kinds of astrophysical phenomena, like extragalactic jets, may require the use of cylindrical coordinates [6]. Spherical coordinates, which are natural choices to investigate the solar coronal plasma, were treated by Hundhausen et al. [7, 8], Tsinganos et al. [9], and Neukirch [10], who have obtained analytical solutions for the ideal MHD equations. Vlahakis and Tsinganos have proposed a systematic method for obtaining general classes of MHD equilibria with azimuthal symmetry [11]. The stability of gravitating plasmas at rest in a magnetic field has been discussed in ref. [12] for ideal plasmas and in ref. [13] for dissipative plasmas.

The equations of ideal MHD can be written in a general form using an arbitrary curvilinear coordinate system, as long as the system has one ignorable coordinate, i.e., a generalized axisymmetry [14]. A related formulation has been given by Kucinski and Caldas [15]. In these cases, the plasma equilibrium is described by two surface functions, namely the transversal magnetic flux and transversal current functions, obeying a generalized version of Grad-Shafranov equation [16].

In this work, we use a general formalism for ideal MHD equilibria so as to add the gravitational field produced by an external source [17]. This amounts to include a third surface

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function, namely the gravitational potential (or the equivalent function for a general external force). In order to close the system of equations, one needs to have some thermodynamical information so as to determine the plasma density. We considered both isothermal and adiabatic processes.

We consider particular cases in rectangular, cylindrical, and spherical geometries, for both isothermal and adiabatic cases. In some of these processes, there are even analytical solutions for the transverse flux function, using particular profiles for the remaining surface functions. This paper is organized as follows: in Section 2, we present the model equations and the general formalism for surface functions leading to a partial differential equation which describes ideal MHD equilibrium with a gravitational field. Sections 3, 4, and 5 present results for rectangular, cylindrical, and spherical geometries, respectively. The final section is devoted to our conclusions.

## 2 MHD Equilibrium Equation with Gravitational Field

We start with the set of ideal MHD equations, describing low-frequency phenomena in a non-viscous conducting fluid without heat conduction (SI units are used) [18]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla p + \mathbf{J} \times \mathbf{B} - \rho \mathbf{g}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}, \tag{6}$$

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \tag{7}$$

where $\rho$, $\mathbf{v}$, $p$, $s$, $\mathbf{J}$, $\mathbf{g}$, $\mathbf{E}$, and $\mathbf{B}$ are, respectively, the mass density, fluid velocity, scalar pressure, specific entropy, current density, external field, electric field, and magnetic induction.

In the case of an external gravitational field, it can be obtained from a scalar potential by $\mathbf{g} = -\nabla \Phi$, where $\Phi$ satisfies Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho_g, \tag{8}$$

where $G$ is the Newtonian gravitational constant and $\rho_g$ is the density of the matter distribution which produces the gravitational field. When describing stellar equilibria and other astrophysical problems where the plasma density $\rho$ is sufficiently large, there is a coupling between (8) and (2), and the set of MHD equations is usually too complicated to be solved analytically. However, if the plasma which equilibrium configuration is to be analyzed has a sufficiently small density, we can decouple the equations, in such a way that the plasma density $\rho$ is different from the density of matter $\rho_g$ which appears in Poisson equation. A classical example is the solar coronal plasma, which has a density $\sim 10^8 cm^{-3}$ much less than the chromosphere $\sim 10^{14}$ cm$^{-3}$, for example. Hence, we can consider the coronal plasma interacting with the gravitational field created by the solar core plasma. We will consider here the latter case, for which the gravitational effect of the plasma itself is negligible.

The static ideal MHD equilibrium equations follow from vanishing time derivatives and velocities in the above system of equations, leading to

$$\nabla p = \mathbf{J} \times \mathbf{B} - \rho \nabla \Phi, \tag{9}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{10}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{11}$$

In the following, we will consider a general configuration described by curvilinear (contravariant) coordinates $(x^1, x^2, x^3)$, and with a given symmetry such that $0 \leq x^3 \leq L$ will be an ignorable coordinate with period $L$. In other words, any physical quantity must not depend on $x^3$ by hypothesis. Moreover, the magnetic axis will be a $x^3$-coordinate curve. It is well-known that a magnetic field having this symmetry and satisfying (11) can be written in the following form [15]

$$\mathbf{B}(x^1, x^2) = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla \Psi + B_3 \frac{\hat{\mathbf{e}}_3}{g_{33}}, \tag{12}$$

where $\hat{\mathbf{e}}_3$ is a covariant basis vector and $g_{33}$ the corresponding element of the covariant metric tensor.

In this representation, we have introduced the transversal flux function $\Psi$, defined by

$$\Psi(x^1, x^2) = \frac{1}{L} \int_S dx^1 dx^3 \sqrt{g}\, \mathbf{B} \cdot \hat{\mathbf{e}}^2, \tag{13}$$

which is the specific (divided by $L$) magnetic flux across a part of the coordinate surface $x^2 = const.$, bounded by the magnetic axis and a $x^3$ coordinate curve. Once $\Psi$ it is known the contravariant magnetic field components are given by:

$$B^1 = -\frac{1}{\sqrt{g}} \frac{\partial \Psi}{\partial x^2}, \tag{14}$$

$$B^2 = \frac{1}{\sqrt{g}} \frac{\partial \Psi}{\partial x^1}. \tag{15}$$

Computing the dot product of (12) with $\nabla \Psi$, there follows that $\mathbf{B} \cdot \nabla \Psi = 0$, i.e., $\Psi$ is a surface function.

The divergence of (10) yields $\nabla \cdot \mathbf{J} = 0$, in such a way that, similarly to the previous case, we can write a representation for the current density

$$\mathbf{J} = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla I + J_3 \frac{\hat{\mathbf{e}}_3}{g_{33}}, \tag{16}$$

where we have defined a transversal current flux $I(x^2, x^2)$, which can be interpreted in terms of the total current flowing through the surface $S_2$ per unit length

$$I(x^1, x^2) = I_{axis} + \frac{1}{L}\int_a^{x^1} dx'^1 \int_0^L dx^3 \sqrt{g} J^2, \tag{17}$$

where $I_{axis} = I(x^1 = a, x^2)$, such that

$$J^1 = -\frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^2} \tag{18}$$

$$J^2 = \frac{1}{\sqrt{g}}\frac{\partial I}{\partial x^1}. \tag{19}$$

Substituting the vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$) in (13) and using Stokes' theorem, we have that, for axisymmetric systems $\Psi(x^1, x^2) = -A_3(x^1, x^2)$. By analogy with Ampére's law (10), there follows that $B_3(x^1, x^2) = -\mu_0 I(x^1, x^2)$ and the magnetic field representation (12) can be rewritten as

$$\mathbf{B} = \frac{\hat{\mathbf{e}}_3}{g_{33}} \times \nabla\Psi - \mu_0 I \frac{\hat{\mathbf{e}}_3}{g_{33}}. \tag{20}$$

Inserting this magnetic field representation into (10) gives, after dotting with the covariant basis vector $\hat{\mathbf{e}}_3$,

$$\mu_0 J_3 = \Delta^*\Psi - \mu_0 I \mathcal{D}, \tag{21}$$

where we have defined a generalized Shafranov operator

$$\Delta^*\Psi = \frac{g_{33}}{\sqrt{g}}\left\{\frac{\partial}{\partial x^1}\left[\frac{\sqrt{g}}{g_{33}}\left(g^{11}\frac{\partial\Psi}{\partial x^1} + g^{12}\frac{\partial\Psi}{\partial x^2}\right)\right] + \frac{\partial}{\partial x^2}\left[\frac{\sqrt{g}}{g_{33}}\left(g^{12}\frac{\partial\Psi}{\partial x^1} + g^{22}\frac{\partial\Psi}{\partial x^2}\right)\right]\right\}, \tag{22}$$

and the following non-orthogonality factor

$$\mathcal{D} = \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\frac{g_{23}}{g_{33}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{13}}{g_{33}}\right)\right], \tag{23}$$

which vanishes for orthogonal coordinate systems.

Moreover, the Lorentz force density $\mathbf{J} \times \mathbf{B}$, after using the representations (20) and (16), reads

$$\mathbf{J} \times \mathbf{B} = -\frac{1}{g_{33}}\left(J_3 \nabla\Psi + \mu_0 I \nabla I\right). \tag{24}$$

Dotting the equilibrium equation (9) with $\hat{\mathbf{e}}_3$, we have $(\mathbf{J} \times \mathbf{B}) \cdot \hat{\mathbf{e}}_3 = 0$. This expression leads, after substituting (24), to the condition $\mathbf{B} \cdot \nabla I = 0$. Hence, $I = I(\Psi)$ is another surface quantity, like in the case without gravitational field. On inserting $\nabla I = I'\nabla\Psi$ back into (24) gives (primes denote differentiation with respect to $\Psi$)

$$\mathbf{J} \times \mathbf{B} = -\frac{1}{g_{33}}\left[J_3 + \frac{1}{2}\mu_0 (I^2)'\right]\nabla\Psi. \tag{25}$$

Substituting (21) and (25) in (9), we obtain

$$\left[\Delta^*\Psi - \mu_0 I \mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)'\right]\nabla\Psi = -\mu_0 g_{33}\nabla p - \mu_0 g_{33}\rho\nabla\Phi. \tag{26}$$

Dotting the force equilibrium equation (9) with $\mathbf{B}$ results

$$\mathbf{B} \cdot \nabla p = -\rho\mathbf{B} \cdot \nabla\Phi. \tag{27}$$

Hence, for a spatially uniform gravitational potential $\mathbf{B} \cdot \nabla p = 0$, from which follows that the magnetic field lines lie on constant pressure surfaces (magnetic surfaces). These surfaces are proven to exist provided there is some form of spatial symmetry of the system. In the usual case, they are nested surfaces with the topology of tori around a degenerate surface (with zero volume) called magnetic axis. However, for a non-uniform gravitational potential, it turns out that $\nabla\Phi$ has a component parallel to $\mathbf{B}$ and thus, while magnetic surfaces may still exist, they are no longer surfaces of constant pressure. In other words, $p$ is no longer a surface quantity (like the volume, for example).

In fact, $p$ will depend on both $\Psi$ and the gravitational potential $\Phi$. Let us suppose that $\nabla\Psi$ and $\nabla\Phi$ be linearly independent, such that we can write

$$\nabla p = \frac{\partial p}{\partial\Psi}\nabla\Psi + \frac{\partial p}{\partial\Phi}\nabla\Phi. \tag{28}$$

which, substituted in (26), yields

$$\left[\Delta^*\Psi - \mu_0 I \mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)' + \mu_0 g_{33}\frac{\partial p}{\partial\Psi}\right]\nabla\Psi + \mu_0 g_{33}\left(\frac{\partial p}{\partial\Phi} + \rho\right)\nabla\Phi = 0. \tag{29}$$

The independence of $\nabla\Psi$ and $\nabla\Phi$ implies that their coefficients in the expression above must vanish identically, what gives us two equilibrium equations,

$$\Delta^*\Psi - \mu_0 I \mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)' + \mu_0 g_{33}\frac{\partial p}{\partial\Psi} = 0, \tag{30}$$

$$\frac{\partial p}{\partial\Phi} = -\rho. \tag{31}$$

If there are no external forces hence $p = p(\Psi)$ and (29) reduces to the Grad-Schlüter-Shafranov equation [14]

$$\Delta^*\Psi - \mu_0 I \mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)' + \mu_0 g_{33} p' = 0. \tag{32}$$

Due to the explicit presence of the plasma density in (31), to advance further, we need a thermodynamic hypothesis. Two possible paths present themselves to the investigation. Firstly, if the plasma obeys the ideal gas equation

$$p = \rho\bar{R}T, \tag{33}$$

where $\bar{R} = (m_e + m_i)k_B$ is the gas constant, with $k_B$ Boltsmann's constant, $m_e$ and $m_i$ respectively the electron and ion masses, and $T(\Psi, \Phi)$ is the temperature. On integrating (31), one obtains

$$p(\Psi, \Phi) = p_0(\Psi) \exp\left[-\int_{\Phi_0}^{\Phi} \frac{d\Phi'}{\bar{R}T(\Psi, \Phi')}\right], \tag{34}$$

where $p_0(\Psi) = p(\Psi, \Phi_0)$. For an isothermal process, $T$ is constant and the above expression leads to the barometric formula:

$$p(\Psi, \Phi) = p_0(\Psi) \exp\left[-\frac{\Phi - \Phi_0}{\bar{R}T}\right]. \tag{35}$$

Differentiating the above expression with respect to $\Psi$ leads to the equilibrium equation in the isothermal case

$$\Delta^*\Psi - \mu_0 I\mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)' + \mu_0 g_{33} p_0' e^{-(\Phi-\Phi_0)/\bar{R}T} = 0. \tag{36}$$

The second type of situation that we can deal with shows up if the plasma satisfies an adiabatic condition

$$p = K\rho^\gamma, \tag{37}$$

where $K = K(s)$ is a constant depending on the plasma entropy and $\gamma = 5/3$ is the ratio of specific heats. Substituting $\rho$ into (31) and integrating

$$p(\Psi, \Phi) = K\left[\left(\frac{p_0(\Psi)}{K}\right)^{1/\eta} - \left(\frac{\Phi - \Phi_0}{\eta K}\right)\right]^\eta, \tag{38}$$

where we defined $\eta = \gamma/(\gamma - 1) = 5/2$.

In the adiabatic case, the equilibrium equation now reads

$$\Delta^*\Psi - \mu_0 I\mathcal{D} + \frac{1}{2}\mu_0^2 (I^2)' + \mu_0 g_{33} p_0' \left(\frac{p_0}{K}\right)^{(1/\eta)-1} \times \left[\left(\frac{p_0}{K}\right)^{1/\eta} - \left(\frac{\Phi - \Phi_0}{\eta K}\right)\right]^{\eta-1}. \tag{39}$$

In both isothermal and adiabatic cases, the equilibrium equations can only be solved if we assume profiles for both $p_0(\Psi)$ and $I(\Psi)$, just like in the usual case without gravitational field. Likewise, the boundary conditions are chosen, in principle, in the same way.

## 3 Solutions in Rectangular Coordinates

Perhaps the simplest application of the formalism so presented is to consider rectangular coordinates ($x^1 = x, x^2 = z, x^3 = y$), for which $g_{33} = 1$ and $\mathcal{D} = 0$. Notice that we consider $y$ to be the ignorable coordinate, such that the system is translationally invariant along it. This situation is compatible with a uniform gravitational field along the $z$-direction: $\mathbf{g} = -g\hat{\mathbf{e}}_z$. The corresponding gravitational potential is $\Phi(z) - \Phi_0 = gz$. In the case of an isothermic plasma, we thus have, from (36), the following equation

$$\Delta^*\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial z^2} = -\frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi} - \mu_0 \frac{dp_0}{d\Psi} e^{-gz/\bar{R}T}. \tag{40}$$

Let us consider the following profiles

$$p_0(\Psi) = \frac{\kappa}{2\mu_0} e^{2\Psi}, \tag{41}$$

$$I^2(\Psi) = I_0^2 = const. \tag{42}$$

Hence, (40) reads

$$\Delta^*\Psi = -\kappa \exp\left(2\Psi - \frac{gz}{\bar{R}T}\right). \tag{43}$$

Making the following change of variables in (43)

$$\psi = \Psi - \frac{gz}{2\bar{R}T}, \tag{44}$$

we obtain $\Delta^*\psi = -\kappa e^{2\psi}$, with the solution [17]

$$\psi(x) = -\ln\cosh(\sqrt{\kappa}x), \tag{45}$$

which, after using (44), is the well-known Kippelhahn-Schlüter solution, describing prominences of the Solar surface, assuming that the distances from the surface are small enough that the Solar magnetic field can be considered nearly uniform along the $z$-direction [4].

The magnetic field components are

$$\mathbf{B} = \left(-\frac{g}{2\bar{R}T}, \sqrt{\kappa}\tanh(\sqrt{\kappa}x), \mp\mu_0 I_0\right). \tag{46}$$

Analogously, the current density components are

$$\mathbf{J} = \left(0, -\frac{\kappa}{\mu_0}\frac{1}{\cosh^2(\sqrt{\kappa}x)}, 0\right). \tag{47}$$

From (46), the magnetic field line equations are

$$\frac{dx}{dy} = \pm\frac{g}{2\bar{R}T\mu_0 I_0}, \tag{48}$$

$$\frac{dz}{dy} = \pm\frac{\sqrt{\kappa}}{\mu_0 I_0}\tanh(\sqrt{\kappa}x), \tag{49}$$

which can be integrated so as to yield the magnetic field lines in the $y = const.$ plane:

$$z = z_0 - \frac{g}{2\bar{R}T\kappa}\ln\left[\frac{\cosh(\sqrt{\kappa}x)}{\cosh(\sqrt{\kappa}x_0)}\right]. \tag{50}$$

A graph of $z$ as a function of $x$ is shown in Fig. 1a for various values of $z_0$. The field lines actually have the curvature characteristic of solar prominences, which are anchored in the Solar photospheric surface.

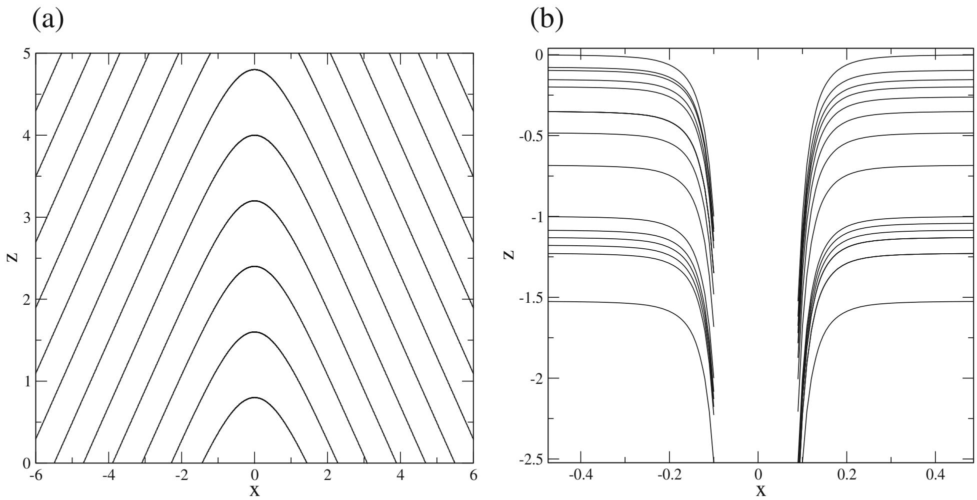


**Fig. 1** Magnetic field lines for the solution of the **a** isothermal and **b** adiabatic MHD equilibrium equation in rectangular coordinates. We indicate normalized coordinates in the axes

Another solution in rectangular coordinates exists for the adiabatic equation (39) which, in rectangular coordinates reads

$$\Delta^*\Psi = -\mu_0 \frac{dp_0}{d\Psi}\left(\frac{p_0(\Psi)}{K}\right)^{(1/\eta)-1}\left[\left(\frac{p_0(\Psi)}{K}\right)^{1/\eta} - \frac{gz}{\eta K}\right]^{\eta-1}. \tag{51}$$

Let us consider a nonlinear profile for the pressure and a constant current function in the form

$$p_0(\Psi) = K\Psi^\eta, \tag{52}$$

$$I^2(\Psi) = I_0^2, \tag{53}$$

in such a way that

$$\nabla^*\Psi = -\mu_0 K\eta\left(\Psi - \frac{gz}{\eta K}\right)^{\eta-1}. \tag{54}$$

Nonlinear pressure profiles have been previously chosen in a study on a special class of time-dependent solutions of the ideal two-dimensional MHD equilibrium equations [19].

As in the previous case, we make a change of variables,

$$\psi = \Psi - \frac{gz}{\eta K}, \tag{55}$$

which turns (54) into $\Delta^*\psi = -\mu_0 K\eta\psi^{\eta-1}$. On assuming that $\psi$ depends only on $x$, we obtain for the above equation the following solution

$$\psi(x) = \left[\frac{-2}{(2-\eta)^2\mu_0 K}\right]^{1/(\eta-2)} x^{2/(2-\eta)} \equiv cx^{2/(2-\eta)}, \tag{56}$$

such that

$$\Psi(x, z) = \left[\frac{-2}{(2-\eta)^2\mu_0 K}\right]^{1/(\eta-2)} x^{2/(2-\eta)} - \frac{gz}{\eta K}. \tag{57}$$

The magnetic field components related to this solution are

$$\mathbf{B} = \left(-\frac{g}{\eta K}, \mp\mu_0 I_0, \frac{2c}{2-\eta}\,x^{\eta/(2-\eta)}\right), \tag{58}$$

and the magnetic field line equation can be integrated in the $y = const.$ plane as before, giving

$$z(x) = z_0 - \frac{\eta K c}{g(2-\eta)^2}\left(x^{2/(2-\eta)} - x_0^{2/(2-\eta)}\right). \tag{59}$$

The magnetic field lines are represented as graphs of $z$ *vs.* $x$ in Fig. 1b, also showing a downward bending caused by the gravitational field, but they do not represent the same type of arcade configurations as in the isothermal case, since the field lines asymptote to infinity as $x \to 0$ due to the power-law behavior expressed by the components in (58).

## 4 Solutions in Cylindrical Coordinates

A certain class of problems with translational symmetry can be solved by using a cylindrical coordinate system: $(x^1, x^2, x^3) = (r, \theta, z)$, where $z$ is an ignorable coordinate, i.e., surface quantities can depend at most on $r$ and $\theta$. Moreover $g_{33} = 1$, $g = r^2$ and $\mathcal{D} = 0$. The MHD equilibrium equation (in the isothermic case) reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta^2} = -\mu_0\frac{dp_0}{d\Psi}e^{-(\Phi-\Phi_0)/\bar{R}T} - \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi}. \tag{60}$$

One problem which fits into this model is the gravitational effect of an infinite mass line with linear density $\lambda$, aligned with the $z$-axis, for which the gravitational field is $\mathbf{g}(r) = -(2G\lambda/r)\,\hat{\mathbf{e}}_r$, which can be derived from the gravitational potential

$$\Phi(r) = \Phi_0 + 2G\lambda \ln\left(\frac{r}{r_0}\right). \tag{61}$$

where $r_0$ is a characteristic length.

Substituting (61) into (60)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta^2} = -\mu_0\frac{dp_0}{d\Psi}\left(\frac{r}{r_0}\right)^{\xi} - \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi}. \tag{62}$$

where we defined

$$\xi = \frac{2G\lambda}{\bar{R}T}, \tag{63}$$

which is the ratio between gravitational and thermal energy.

We can rewrite (62) after choosing the following profiles

$$p_0(\Psi) = A(\Psi - \Psi_0), \tag{64}$$
$$I^2(\Psi) = I_0^2, \tag{65}$$

such that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) = -\mu_0 A r_0^{\xi} r^{1-\xi}, \tag{66}$$

where we supposed that the system also exhibits azimuthal symmetry, yielding the solution

$$\Psi(r) = \Psi_0 - \frac{\mu_0 A r_0^{\xi}}{(2-\xi)^2}\left(r^{2-\xi} - r_0^{2-\xi}\right), \tag{67}$$

valid for $\xi \neq 2$. The magnetic surfaces in this case are cylinders of radius $r$, and the magnetic axis is located at $r = 0$.

Once $\Psi$ is known, the magnetic field components are given by

$$\mathbf{B} = \left(0, -\frac{\mu_0 A r_0^{\xi}}{2-\xi} r^{1-\xi}, \mp\mu_0 I_0\right), \tag{68}$$

such that the magnetic field line equations can be integrated to give

$$r = const., \qquad \theta(z) = \pm\frac{A}{(2-\xi)I_0}\left(\frac{r_0}{r}\right)^{\xi}, \tag{69}$$

in such a way that the field lines have helical paths on the flux surfaces, their winding number depending on the radius, i.e., there is a magnetic shear involved (in the form of a monotonic twist radial profile) (Fig. 2). Moreover, the magnetic shear depends on the gravitational field through the parameter $\xi$, which is the ratio between the gravitational and thermal energies. When $\xi > 1$, the former dominates the latter, the situation changing if $\xi < 1$. The pitch of the helices decreases as $\xi$ increases, meaning a stronger gravitational effect.

Now let us consider the adiabatic MHD equilibrium equation (39) in cylindrical coordinates, viz.

$$\Delta^*\Psi = -\frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0\frac{dp_0}{d\Psi}\left(\frac{p_0}{K}\right)^{(1/\eta)-1} \times\left[\left(\frac{p_0}{K}\right)^{1/\eta} - \frac{2G\lambda}{\eta K}\ln\left(\frac{r}{r_0}\right)\right]^{\eta-1}, \tag{70}$$

Inserting in the above expression the same profiles used in the rectangular case, (64), after a similar calculation there results

$$\nabla^*\psi = -\mu_0 K\eta\psi, \tag{71}$$

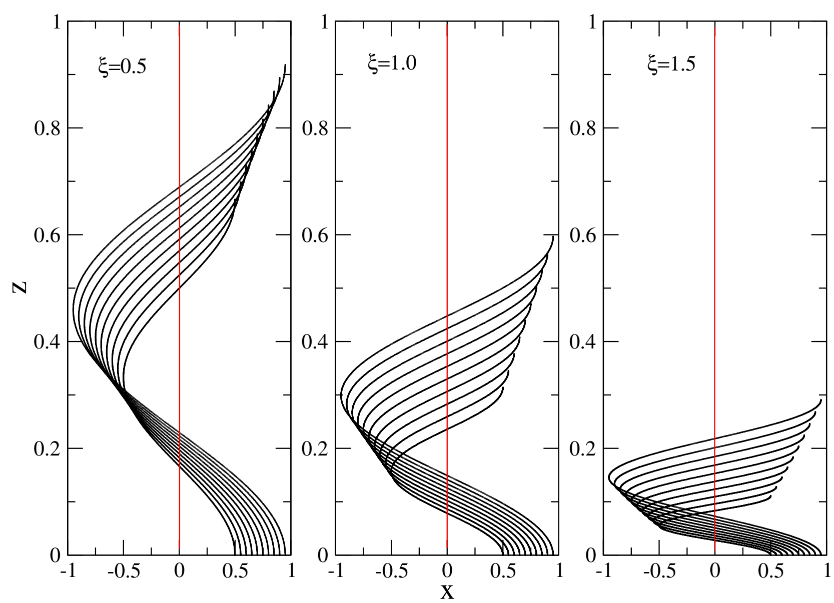


**Fig. 2** Magnetic field lines for the solution of the isothermal MHD equilibrium equation in cylindrical coordinates, for $\xi = 0.5$ (**a**), 1.0 (**b**), and 1.5 (**c**). We indicate normalized coordinates in the axes. The *red line* stands for the source of the gravitational field

where we have defined

$$\psi = \Psi - \frac{2G\lambda}{\eta K}\ln\left(\frac{r}{r_0}\right). \tag{72}$$

If we, in addition, also assume that $\psi$ does not depend on $\theta$, the above equation can be analytically integrated, yielding

$$\psi(r) = \left[\frac{-\mu_0 K(2-\eta)^2}{2}\right]^{1/(2-\eta)} r^{2/(2-\eta)} \equiv d r^{2/(2-\eta)}, \tag{73}$$

such that

$$\Psi(r) = d r^{2/(2-\eta)} + \Lambda \ln\left(\frac{r}{r_0}\right), \tag{74}$$

where we defined the nondimensional parameter

$$\Lambda = \frac{2G\lambda}{\eta K}. \tag{75}$$

The magnetic field components obtained from (74) allows for an exact integration of the corresponding magnetic field line equations, yielding

$$\theta(r,z) = \left(\frac{2d}{(2-\eta)\mu_0 I_0}\right) r^{-2/(2-\eta)} z + \frac{2G\lambda}{\eta K \mu_0 I_0}\frac{z}{r^2}, \tag{76}$$

and are plotted in Fig. 3 for different values of the parameter $\Lambda$. Since $\Psi$ depends only on $r$ the magnetic surfaces are cylinders but the field lines spiral with larger pitch when compared with the isothermal case. Moreover, the gravitational effect (increasing $\Lambda$) bends more the field lines, and the pitches are correspondingly shorter.

## 5 Solution in Spherical Coordinates

A wide class of problems of interest in astrophysics deals with spherically symmetric configurations, for which we use spherical coordinates $(x^1, x^2, x^3) = (r, \theta, \phi)$, with $\phi \in [0, 2\pi)$ as the ignorable coordinate. In addition, we have $g_{33} = r^2 \sin^2\theta$ and $\mathcal{D} = 0$. The isothermic MHD equilibrium equation is

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = \mu_0 r^2 \sin^2\theta \frac{dp_0}{d\Psi} e^{-(\Phi-\Phi_0)/\bar{R}T} - \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi}. \tag{77}$$

The simplest spherically symmetric problem is the gravitational potential of a sphere of mass $M$ and radius $a$, with a uniform density

$$\rho(r) = \begin{cases} \rho_0 = \frac{3M}{4\pi a^3}, & \text{if} \quad 0 \le r \le a, \\ 0, & \text{if} \quad r > a \end{cases},$$

which is given by

$$\Phi(r) = \begin{cases} -GM\frac{3a^2 - r^2}{2a^3}, & \text{if} \quad 0 \le r \le a, \\ -GM\frac{1}{r}, & \text{if} \quad r > a \end{cases}. \tag{78}$$

Considering only the plasma inside the sphere and defining

$$\Phi_0 = \frac{3GM}{2a}$$

the equation (77) becomes

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = \mu_0 r^2 \sin^2\theta \frac{dp_0}{d\Psi} e^{-\varpi(r^2/a^2)} - \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi}. \tag{79}$$

where another nondimensional parameter has been introduced:

$$\varpi = \frac{GM}{2\bar{R}Ta}. \tag{80}$$

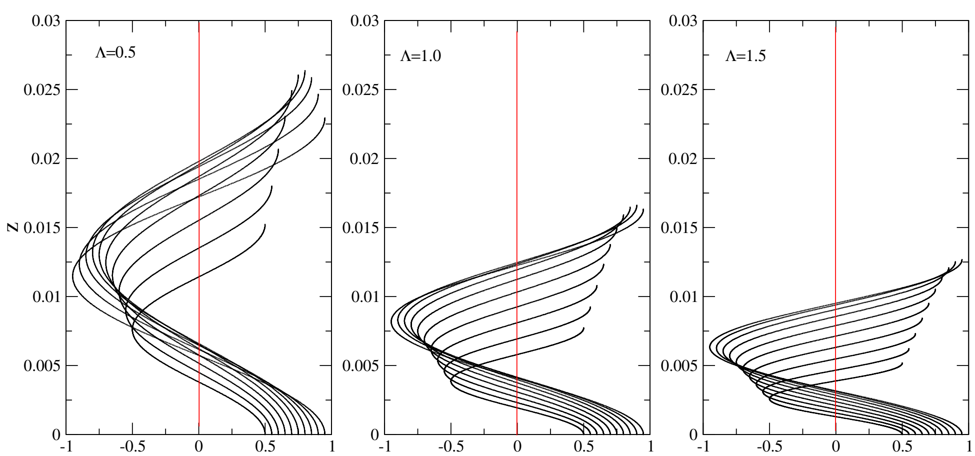


**Fig. 3** Magnetic field lines for the solution of the adiabatic MHD equilibrium equation in cylindrical coordinates, for $\Lambda = 0.5$ (**a**), 1.0 (**b**), and 1.5 (**c**). We indicate normalized coordinates in the axes. The *red line* stands for the source of the gravitational field

We choose the same profiles as in the cylindrical isothermal case, namely (64), and obtained

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = -\mu_0 A r^2 \sin^2\theta e^{-\varpi(r^2/a^2)}. \tag{81}$$

We solve the above equation using separation of variables

$$\Psi(r,\theta) = f(r)\sin^2\theta, \tag{82}$$

where the radial function satisfies the following differential equation

$$\frac{d^2 f}{dr^2} - \frac{2f}{r^2} = -\mu_0 A r^2 e^{-\varpi r^2/a^2}. \tag{83}$$

Using the nondimensional variables $x = r/a$ and $y = f/\mu_0 A$, this equation reads

$$\frac{d^2 y}{dx^2} - \frac{2y}{x^2} = -a^2 x^2 e^{-\varpi x^2}, \tag{84}$$

which can be solved using standard Green's function techniques.

The corresponding homogeneous equation is of Sturm-Liouville form with integrating factor $p(x) = 1$ and has two linearly independent solutions, namely

$$y_1(x) = x^2, \qquad y_2(x) = x^{-1}, \tag{85}$$

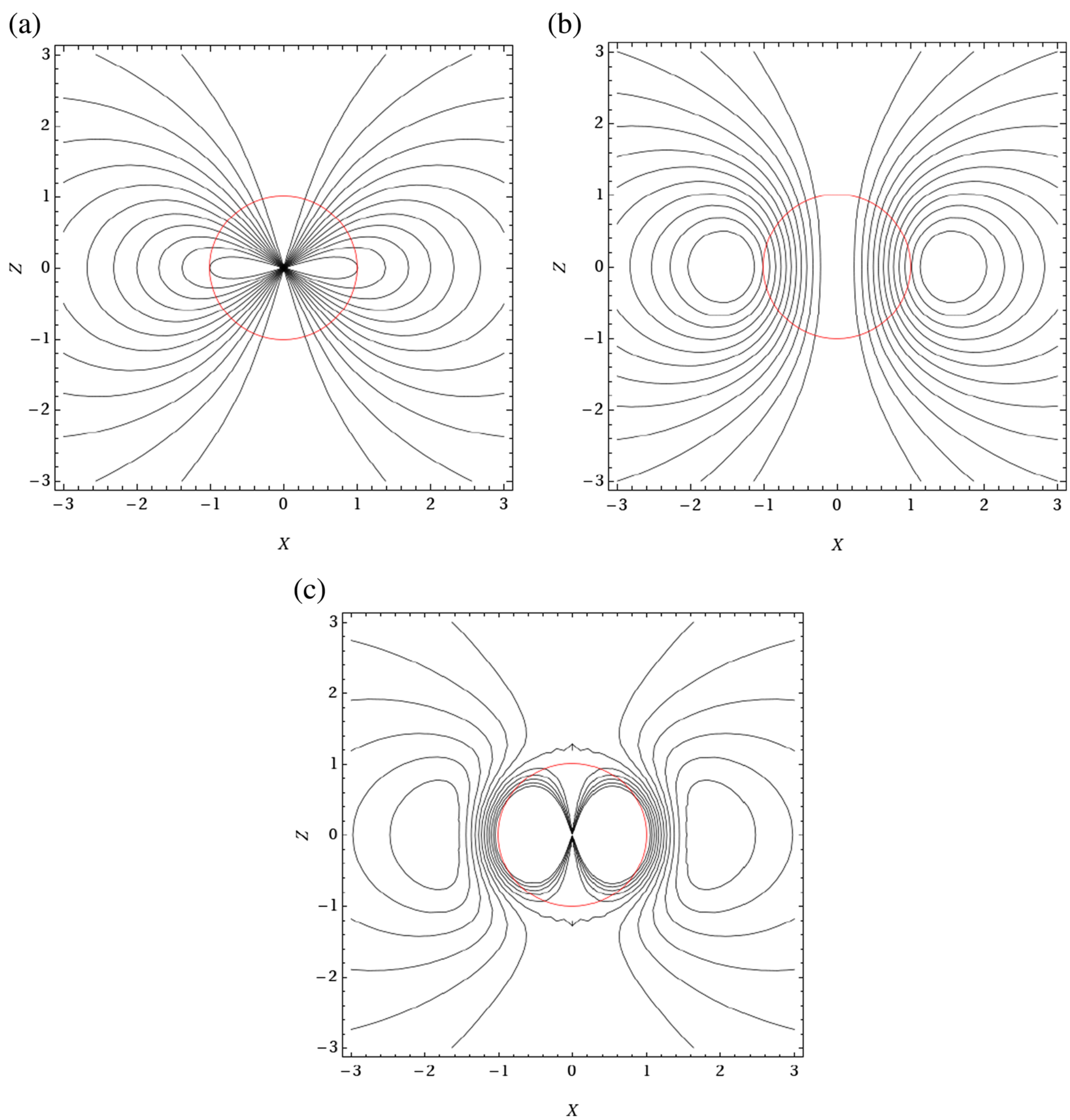


**Fig. 4** Magnetic field lines for the solution of the adiabatic MHD equilibrium equation in spherical coordinates, for $\varpi = 0.5$ (**a**), 1.0 (**b**), and 1.5 (**c**). We indicate normalized coordinates in the axes. The *red circle* stands for the source of the gravitational field

since their Wronskian $W(x) = -3$ is nonvanishing. The Green's function is therefore [20]

$$G_1(x,\xi) = -\frac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} = \frac{x^2}{3\xi}, \qquad (0 \le x \le \xi), \quad (86)$$

$$G_2(x,\xi) = -\frac{y_2(x)y_1(\xi)}{p(\xi)W(\xi)} = \frac{\xi^2}{3x}, \qquad (\xi \le x < \infty), \quad (87)$$

such that the solution of the inhomogeneous equation (84), with the source term $F(x) = a^2x^2e^{-\varpi x^2}$, is

$$\begin{aligned} y(x) &= \int_0^x d\xi\, G_2(x,\xi)F(\xi) + \int_x^\infty d\xi\, G_1(x,\xi)F(\xi) \\ &= \frac{a^2\sqrt{\pi}}{8\varpi^{5/2}}\,\frac{\mathrm{erf}(\sqrt{\varpi}x)}{x} \\ &\quad -\frac{a^2}{6\varpi}\,e^{-\varpi x^2}\left[\frac{3}{2} + x^2(\varpi - 1)\right], \end{aligned} \quad (88)$$

and the flux function reads

$$\Psi(r,\theta) = \mu_0 A y(r)\sin^2\theta. \quad (89)$$

The magnetic field lines corresponding to this solution are depicted in Fig. 4. When the parameter $\varpi$ is less than the unity, i.e., the thermal energy is larger than the gravitational energy, the magnetic field has a dipolar structure (Fig. 4a). If the converse is true ($\varpi > 1$), then the magnetic field lines form different closed loops inside and outside the spherical source of gravitational field (Fig. 4c). The outside loops have the same dipolar structure as before, whereas the inside loops are closed. Therefore, in this case, we can regard the spherically symmetric plasma as being effectively confined by the gravitational field, as in a magnetic star.

## 6 Conclusions

Exact solutions for ideal MHD equilibrium equations are always welcomed since they can be used to benchmark numerical methods of solution. This is known to be useful in fusion plasmas, for which equilibrium numerical codes sometimes present challenging problems of stability and convergence. In the case of a gravitational field, such solutions can also be used with this purpose, at least in the simpler case of problems here treated, namely, when the plasma itself has a small density, such that the gravitational field is generated by another body. In this case, Poisson equation is decoupled from the ideal MHD equations, allowing for analytical solutions with selected profiles of both pressure and current function.

In this work, we deduced a general ideal MHD equilibrium equation in presence of such external gravitational field, with the assumption that there is an ignorable coordinate, what reduces the problem to two dimensions effectively. The presence of plasma density imposes the use of a thermodynamical assumption, and we considered both isothermal and adiabatic cases. For more detailed studies of stellar equilibria, however, a politropic process would be a better choice. We present an exact solution for the MHD equilibrium equation in rectangular coordinates in the adiabatic case. It has features similar to the corresponding isothermal equation, whose solution dates back to the pioneer work of Kippenhahn and Schluter, namely the bending of magnetic field lines caused by the gravitational field. By Alfvén theorem, since the plasma clings to the field lines in the ideal case, such solution can describe solar prominences.

We also present exact solutions for the equation in cylindrical coordinates in both isothermal and adiabatic cases. The gravitational field with such symmetry can be found in models of astrophysical jets. Finally, we present an exact solution for the equation in spherical coordinates in the isothermal case. In the case of stronger gravitational energy (compared to the thermal energy), the field lines are closed inside the spherical source of the gravitational field, which brings about a gravitational confinement of a magnetized plasma. Unfortunately, we did not find such solution for the adiabatic case, but it is probable that this could be possible by making a convenient separation of variables, such that part of the solution be analytical and the other can be numerically obtained as a two-point boundary value problem. We think that the solutions we obtained can be used in a variety of astrophysical problems with such symmetry such as globular nebulae and other related objects.

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