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Unstable dimension variability structure in the parameter space of coupled Hénon maps

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A R T I C L E I N F O

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A B S T R A C T

Coupled maps have been investigated to model the applications of periodic and chaotic dynamics of spatially extended systems. We have studied the parameter space of coupled Hénon maps and showed that attractors possessing unstable dimension variability (UDV) appear for parameters neighbouring the so called shrimp domains, representing parameters leading to stable periodic behaviour. Therefore, the UDV should be very likely to be found in the same large class of natural and man-made systems that present shrimp domains.

1. Introduction

In the early 1960s, Lorenz proposed three coupled nonlinear differential equations to model atmospheric convection [1]. The Lorenz equations have attracted great attention due to their interesting dynamical solutions, for instance, a chaotic attractor [2,3]. Inspired by the Lorenz attractor, Hénon introduced a map from the Poincaré section of the Lorenz equations [4], the Hénon map, which despite being simpler than differential equations, still preserve most of the relevant features of the solutions of the Lorenz system. Hénon's map behaviour can be set by its parameters. A global two dimensional visualisation of the map's behaviour as a function of two parameters is called the parameter space. It indicates by colours the type of solutions found in the system for those two parameters. Same colour regions represent an open set of parameters that lead the system to the same behaviour. These open sets form domains in the parameter space, and appear organised in a very regular way. Gallas, studying the Hénon and another unidimensional map, Gallas [5,6] reported the existence of shrimps-shaped domains that appear along special directions and organised in a self-similar way representing parameters for which the maps present periodic stable behaviour. Many other works have reported shrimp-shaped domains in parameter spaces of other mathematical models, for instance, Hoff and collaborators [7] verified the isoperiodic structures in a four-dimensional Chua model with smooth nonlinearity. Periodic windows were observed in mathematical models of two-gene for gene expression and regulation [8]. It was also verified the existence of shrimp-shaped domains in an experimental

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Chua's circuit [9]. Medeiros and collaborators showed that periodic domains arose in the parameter space of an impact oscillator [10].

In this work, we study coupled Hénon maps. Coupled maps networks are one of the most investigated types of spatially extended systems where both space and time are discrete variables while the state variable is continuous. The system is formed by a local dynamical unit interacting with other units by means of a coupling architecture. Coupled maps networks have been considered in mathematical models to study biological neuronal networks [11–13], image encryption [14], secure communication [15], and spatiotemporal chaos [16].

Here, we aim at finding parameter regions responsible to make a coupled network of Hénon maps to present unstable dimension variability (UDV). The UDV describes a situation in which the dimension of the unstable and stable tangent spaces are not constant over the chaotic set [17,18]. Unstable dimension variability was introduced by Abraham and Smale by means of a two-dimensional continuously differentiable map [19]. If an attractor contains periodic orbits with different number of unstable directions, the attractor has UDV [18]. In 1997, Kostelich and collaborators [17] analysed the torus map, a simple two-dimensional map, that displays UDV. They showed analysis and simulations to illustrate the dynamics behaviours of systems with UDV. Noninvertible two-dimensional map can have an invariant set exhibiting UDV [18]. UDV in a three-dimensional map was observed by Dawson [20]. In this work, we consider coupled Hénon maps to study higher-dimensional dynamical systems. We build a network where the maps are globally coupled, known as all-to-all coupling. It is still allusive how periodic windows appear in coupled systems as the dimension of the system increases. This paper however shows that despite the system studied has dimension 6, these windows are still present appearing nearby regions with attractors possessing UDV.

Many mathematical models exhibit UDV phenomenon such as the double rotor [21], coupled chaotic maps [22], and periodically forced drift waves [23]. In the UDV, there is the absence of shadowing trajectories, resulting in that a long numerical trajectory may not be associated with any mathematically true solution [24,25]. This way, it is important to investigate UDV in chaotic systems to know if the computed solution is close to a true solution, namely if there is UDV, simulations will produce chaotic pseudo-solutions. Due to the fact that the trajectory moves in regions with different unstable dimensions, the values of the Lyapunov exponents oscillate between negative and positive values [26]. Dawson and collaborators [27] identified UDV in computer simulations of a double rotor. They verified that the second largest exponent fluctuates about zero. Therefore, we use the finite time Lyapunov exponents as a diagnostic of UDV [28].

This article is organised as follows: in Section 2 we present the model of coupled Hénon maps. In Section 3 we study domains of unstable dimension variability in the parameter space. In the last section, we draw the conclusions.

2. The mathematical model

The Hénon map is a two-dimensional discrete time dynamical system given by

$$\begin{aligned} x_{n+1} &= a - x_n^2 + b y_n, \\ y_{n+1} &= x_n, \end{aligned} \tag{1}$$

where x_n and y_n are the continuous states at discrete time-n ($n = 0, 1, 2, \dots$), a and b are positive parameters of the system. For the Hénon map over a range of a and b values it is possible to obtain chaotic, intermittent, or periodic behaviour. The chaotic behaviour can be identified when the largest Lyapunov exponent is positive [29]. The kth Lyapunov exponent is defined as

$$\lambda_k(\mathbf{x}_0, n) = \frac{1}{n} \ln(||\mathbf{Df}^n(\mathbf{x}_0)\mathbf{u_k}||), \tag{2}$$

where $\mathbf{Df}^n$ is the Jacobian matrix of the n-time iterated map evaluated at point $\mathbf{x}_0$ and $\mathbf{u_k}$ is the eigenvector corresponding to the kth eigenvalue of the Jacobian matrix. For infinite-times, we have the usual Lyapunov exponent

$$\lambda_k = \lim_{n\to\infty} \lambda_k(\mathbf{x}_0, n), \tag{3}$$

where the value is the same for almost every initial condition $\mathbf{x}_0$. To identify UDV, we consider the finite time-n Lyapunov exponent, calculated by making n in Eq. (2) small and finite.

Fig. 1(a) shows the parameter space for the Hénon map with the largest Lyapunov exponent represented by the colour scale. We can see periodic domains in black and grey scale regions immersed in chaotic regions ($\lambda_1 > 0$), and in the white region the trajectories diverge to infinity. The periodic structures, known as shrimp-shaped, can also be identified by the Kaplan–Yorke dimension (Fig. 1(b)) that is defined as

$$D = \begin{cases} 0 & \text{if there is no such } m, \\ m + \frac{1}{|\lambda_{m+1}|} \sum_{k=1}^{m} \lambda_k & \text{if } m < N, \\ N & \text{if } m = N, \end{cases} \tag{4}$$

where m is the greatest integer for which $\sum_{k=1}^{m} \lambda_k \geq 0$ and N is the dimension of the system.

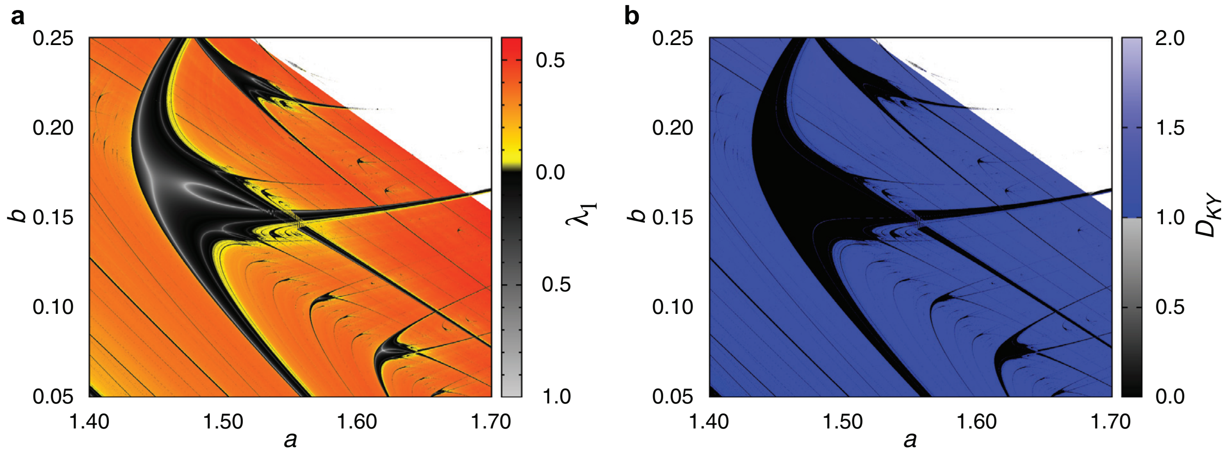

Fig. 1. Parameter space of the Hénon map, for the parameters a and b, where the colour code (given by the sidebar) represents the values of (a) the largest Lyapunov exponent and (b) the Kaplan–Yorke dimension.

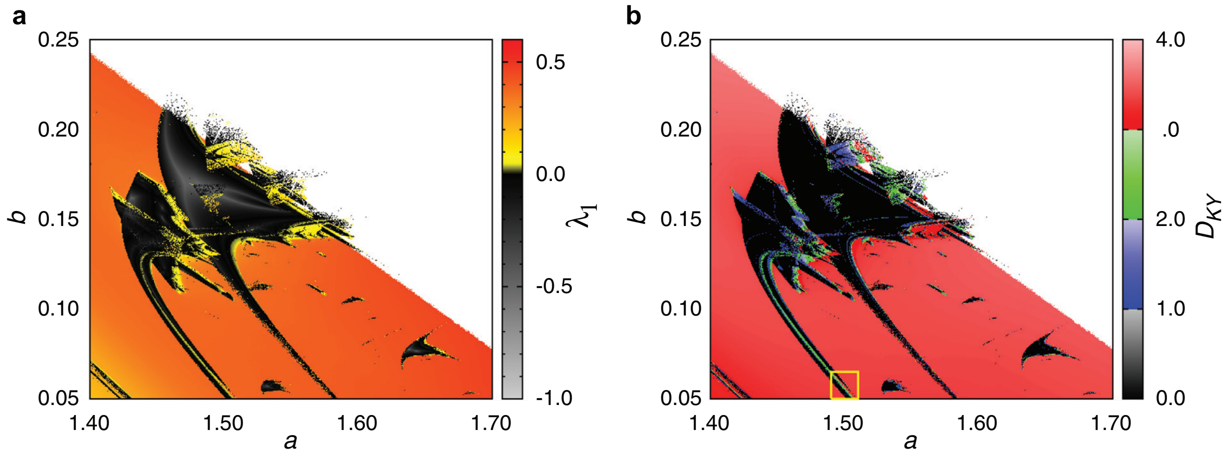

Fig. 2. Parameter space for parameters a and b for a system with 3 Hénon maps where the colour code represents the values of (a) the largest Lyapunov exponent and (b) the Kaplan–Yorke dimension. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

In this work, we consider the following network of coupled Hénon maps given by

$$x_{n+1}^{(i)} = a - x_n^{(i)2} + by_n^{(i)} + \varepsilon(x_n^{(i-1)} + x_n^{(i+1)}),$$
$$y_{n+1}^{(i)} = x_n^{(i)}, \quad (5)$$

where each map is labelled by the super-index i ($i = 1, 2, 3$) and ε is the coupling strength. Fig. 2(a) exhibits the largest Lyapunov exponent for a system with 3 Hénon maps and coupling strength equal to 0.0219. The size of the white region, in that the trajectories diverge to infinity, is larger than the region one would have obtained if a parameter space had been constructed considering a single map. In Fig. 2(b) we calculate the Kaplan–Yorke dimension for the coupled maps, where we can see that the maximum value is equal to 4 and there are many small regions with values in the interval between 1 and 3. For periodic attractors, $D = 0$, and for chaotic ones $D > 0$.

3. Results and discussions

Fig. 3(a) shows a magnification of a portion of the parameter space of Fig. 2(b). In Fig. 3(b), we can see the stable periodic orbits, for $D \approx 0$. The parameter region in that we find stable periodic behaviour is called from here on periodic window.

In our simulations, the periodic windows in Fig. 3(a) and (b) exhibits stable period doubling bifurcations when the parameter a increases from inside the periodic windows to the chaotic region, and when a is decreased from the inside of the periodic window to the chaotic region there is a crisis induced intermittency route to chaos. Comparing Fig. 3(a) with (b), we can see a green region (Fig. 3(a)) close to a periodic window on the side of the period doubling cascade route to chaos.

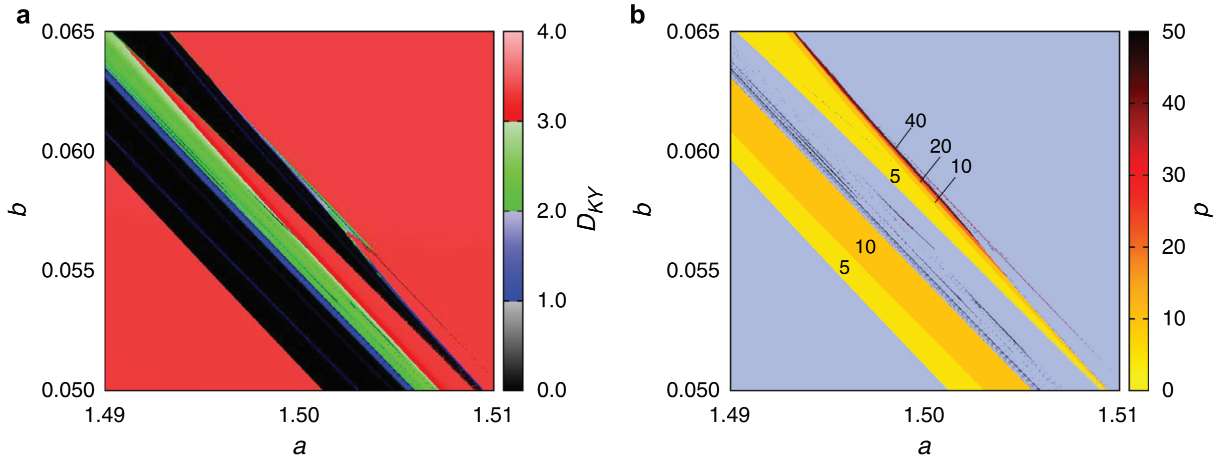

Fig. 3. Magnification of the parameter space marked by the yellow rectangle in Fig. 2(b), where the colour code in (a) represents the Kaplan–Yorke dimension, and in (b) represents the period of stable periodic orbits. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

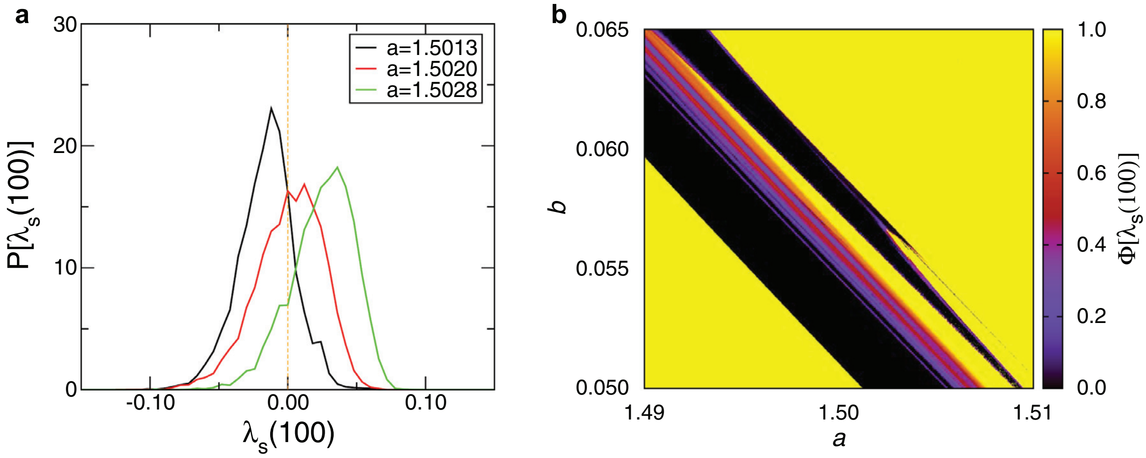

Fig. 4. (a) Histogram of the largest transversal finite time Lyapunov exponents ($\lambda_s(100)$) for $a = 1.5013$ (black line), $a = 1.5020$ (red line), and $a = 1.5028$ (green line). (b) Parameter space where the colour bar represents the fraction of positive values of $\Phi(\lambda_s)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Regarding the green region, we calculate the finite-time Lyapunov exponents and consider one of then, denoted by λ_s to characterise the presence of UDV. An attractor presents UDV when at least one finite time Lyapunov exponent fluctuates around zero, generating unshadowable trajectories [27]. This occurs due to the fact that the trajectory goes to regions with stable and unstable directions embedded within a chaotic attractor [30].

Fig. 4(a) shows some histograms for λ_s calculated for a finite-time $n = 100$ obtained for different values of the parameter a. The histograms have a Gaussian-like shaped with different variances. For $a = 1.5013$ (black line) UDV occurs due to the small fraction of positive values of λ_s. When the parameter a is increased from 1.5020 (red line) to 1.5028 (green line), the fraction of positive values of λ_s increases. This fraction of positive values continues to increase until the complete disappearance of the UDV. In this work, we show results considering only $n = 100$, adequate for UDV analyses. We try simulations for different values of n. For instance, for values $n = 1000$ and $n = 10$, determination of the values of a and b corresponding to where the UDV appears is compromised. Low n values provides arbitrarily close a and b values that produce substantially different Φ values in Eq. (6). Large n values make Φ to be close to zero for large ranges of a and b.

To make our following analysis more rigorous, we calculated the fraction of positive values of λ_s by

$$\Phi = \frac{\int_0^\infty P(\lambda_s)\, d\lambda_s}{\int_{-\infty}^\infty P(\lambda_s)\, d\lambda_s}, \tag{6}$$

UDV is characterised by having $0 < \Phi < 1$. In Fig. 4(b) the colour code represents Φ. We can see that the region of UDV is in the green region illustrated in Fig. 3(a).

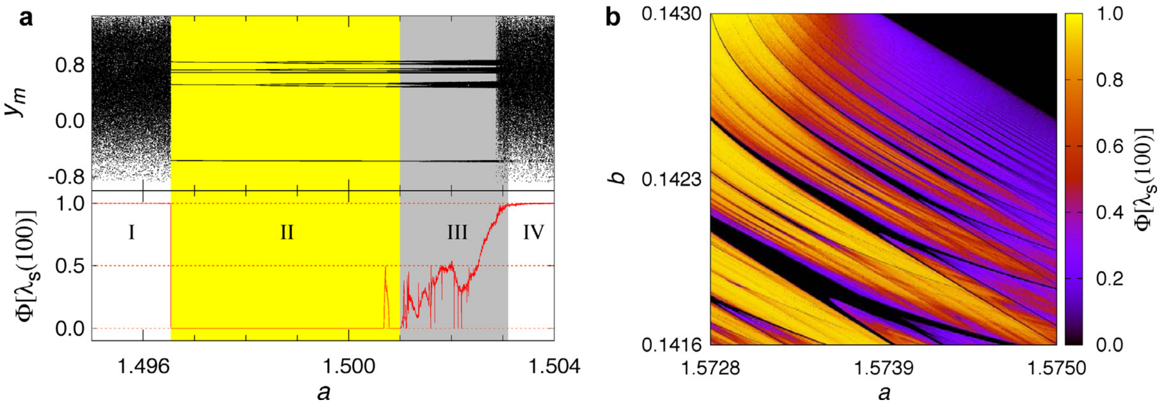

Fig. 5. (a) Bifurcation diagram and the fraction of positive Lyapunov exponents for $b = 0.054$, where I and IV represents the chaotic region, the period-doubling bifurcation is showed in the region II, and III exhibits the region of the UDV. (b) Parameter space for the fraction of positive $\Phi(\lambda_s)$ in scale colour.

To have a better visualisation of the dynamics behaviour of our system while presenting UDV, we calculate a bifurcation diagram for $b = 0.054$ and a in a sub range of Fig. 4(b), $a \in [1.495, 1.504]$ (Fig. 5(a)). The chaotic behaviour is showed in the regions I and IV, while the region II exhibits period-doubling bifurcation. There is a crisis route to chaos from region II to region I. In the region III $\Phi < 1$, indicating that in this region the coupled Hénon maps present UDV. Region III has parameter boundaries delimited to the left by a merging band crisis and to the right by an interior crisis.

To show that UDV appears always side by side of periodic windows, in Fig. 5(b) we plot the parameter space for a large range of parameters. One can see that they are always close to the periodic windows open regions for which $\Phi < 1$, indicating that UDV is likely to be found in our system, and following the same self-similar structure of periodic windows. Due to the fact that shrimps can be found over large parameter domains, it is also possible to find UDV over large domains in the parameter space.

4. Conclusions

We have studied the parameter space of coupled Hénon maps. We paid special attention to the Lyapunov exponent that oscillates around zero, an evidence that the corresponding attractor presents unstable dimension variability, a condition that prevents one from obtaining true trajectories from numerical simulations. This work reveals that parameters that lead to UDV appears alongside periodic windows. An heuristic explanation for this result can be given by motivating that shrimps appears by a tangent bifurcation, taking the last surviving positive Lyapunov exponent to have an asymptotic averaged null value. However, finite estimations of the largest Lyapunov exponent should reflect its time fluctuations about the zero value. The extension of these results to other similar systems should be investigated. Nevertheless, a rigorous confirmation of this properties is still an open question. Therefore, the same scaling laws describing the global and local appearance of periodic windows can be used to describe UDV domains in the parameter space. Finally, this work shows that UDV is likely to be found in coupled systems.

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