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Robust tori-like Lagrangian coherent structures

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ABSTRACT

In general the term "Lagrangian coherent structure" (LCS) is used to make reference about structures whose properties are similar to a time-dependent analog of stable and unstable manifolds from a hyperbolic fixed point in Hamiltonian systems. Recently, the term LCS was used to describe a different type of structure, whose properties are similar to those of invariant tori in certain classes of two-dimensional incompressible flows. A new kind of LCS was obtained. It consists of barriers, called robust tori that block the trajectories in certain regions of the phase space. We used the Double-Gyre Flow system as the model. In this system, the robust tori play the role of a skeleton for the dynamics and block, horizontally, vortices that come from different parts of the phase space.

1. Introduction

Lagrangian coherent structures (LCS) are objects within the flow that govern the transport and form a robust structure for the dynamics. The concept and terminology for the LCS were first introduced by Haller [1,2] and Haller and Yuan [3] who presented mathematical criteria for the existence of finite time attracting and repelling material surfaces (i.e., finite-time hyperbolic invariant manifolds) in time-dependent flows. This is a time-dependent analog of stable and unstable manifolds from a hyperbolic fixed point in Hamiltonian systems [4]. This was the first kind of LCS reported in the literature [1–3], and some applications have also been reported, as the study of vortex on the polar stratospheric [5], dispersion of pollution in oceans [6], vortex rings created from a free-swimming *Aurelia aurita* jellyfish [7] and turbulent transport in magnetized fusion plasmas [8].

Another kind of known LCS, called Invariant-Torus-Like LCS, consists in a time-dependent analog of the Shearless Torus [9], which appears in Hamiltonian systems with non-monotonic rotation numbers, studied recently by Beron-Vera et al. [10]. They analyzed a system that simulates the transport barrier near the core of the polar night jet, in the Earth's lower and middle stratosphere in the austral winter and early spring. In this approach the barrier traps the ozone-depleted air inside the ozone hole [9].

In this paper, we present a new kind of LCS, which is a time-dependent analog of the barriers called robust tori (RT) [11–13] that appear in Hamiltonian systems, in which the perturbations algebraically vanish at some defined positions, blocking the transport along the phase space.

In contrast with the KAM tori that, depending on their winding number, may persist only for some sufficiently small perturbations, the RT consist of curves in the phase space that remain intact, even for generic perturbations, playing the role of permanent barriers for the dynamics [11–13].

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2. The model

In order to analyze the influence of RT in a (non-integrable) time-dependent Hamiltonian, we choose the Double-Gyre Flow [14] as the model, in which the particle motion in a two-dimensional incompressible flow obeys the equations, $\dot{x} = -(\partial\Psi/\partial y)$ and $\dot{y} = (\partial\Psi/\partial x)$, where the streamfunction $\Psi(x, y, t)$ is the non-autonomous Hamiltonian for this one and a half degree of freedom system. t denotes the time and the pair (x, y) is related with the position of the particle in the fluid. This model should not be seen as the approximate solution to a real fluid flow, but rather a simplification of a double-gyre pattern that occurs frequently in geophysical flows [15]. It can be described by the following streamfunction,

$$\Psi(x, y, t) = A\sin(f(x, t)\pi)\sin(y\pi) \tag{1}$$

where $f(x, t)$ has the form [15],

$$f(x, t) = \varepsilon\sin(\omega t)x^2 + (1 - 2\varepsilon\sin(\omega t))x. \tag{2}$$

Then,

$$\Psi(x, y, t) = A\sin[\pi(\varepsilon x^2\sin(\omega t) + x(1 - 2\varepsilon\sin(\omega t)))]\sin(y\pi). \tag{3}$$

From this equation we obtain the corresponding system of equations of motion

$$\begin{aligned} \dot{x} &= -\pi A\sin(f(x, t)\pi)\cos(y\pi) \\ \dot{y} &= \pi A\cos(f(x, t)\pi)\sin(y\pi)\frac{\mathrm{d}f(x, t)}{\mathrm{d}x}. \end{aligned} \tag{4}$$

The condition to obtain a RT-like LCS occurs when the perturbation, and at least one of the equations of motion, vanish forming straight lines in phase space. These lines behave as dynamical barriers for the trajectories. We note that for $\sin(y\pi) = 0$, the function in Eq. (3) is null even for $\varepsilon \neq 0$. From Eq. (4) we also observe that the motion in the y-direction vanishes, and we have $y =$ constant for any time, leading to infinite barriers in the position where y is an integer number, which is the condition for the existence of robust tori-like LCS in the system.

For $\sin(f(x, t)\pi) = 0$, the barriers can appear in the x-direction in two cases: (1) when $\varepsilon = 0$, the barriers will appear for every integer value of x; and (2) when $\varepsilon \neq 0$, the barriers will appear only at $x = 0$ and $x = 2$, as can be seen from Eqs. (1) and (2).

Since the structure of the equations, as well as the cases to form RT-like LCS are known, the next step are the numerical simulations in order to verify the expected results and, most important, to understand the meaning of the parameters of the equations that describe the motion.

3. Numerical results

Fig. 1 shows the phase space of the system governed by Eq. (3) for $\varepsilon = 0$. This parameter specifies how large the perturbation is, in the dynamical system. The results were obtained using the Runge–Kutta integration method, for the equations of motions shown in Eq. (4). All RT in the y-direction and in the x-direction are intact (the straight lines). On the other hand when $\varepsilon \neq 0$, only robust tori in the y-direction will survive, as well as, the ones located at $x = 0$ and $x = 2$.

Fig. 2 shows the velocity fields, where we fixed all parameters and obtained the plots for different time intervals. We used the same parameters cited in Ref. [14]. The terms $\sin(y\pi)$ and $\cos(y\pi)$ in Eq. (4), introduce a symmetry in the system as the time increases. Looking at Fig. 2(a) we can see the vortices going to the right and, after half a period, the velocity fields repeat this behavior, inverting the vortices' direction along the x-direction, Fig. 2(b). Fig. 2(c) shows the vortices going to the left, until they complete one period, in Fig. 2(d), and start the same motion again.

Now, we introduce the finite time Lyapunov exponent (FTLE) that measures the deviation of neighboring trajectories. It is an important tool because, in time-dependent flows, the instantaneous velocity fields (as shown in Figs. 2), in general, do not reveal all information about the trajectories. On the other hand the FTLE accounts for the integrated effect of the flow because it is derived from particle trajectories, and thus it is a more efficient tool to analyze the transport behavior. It measures how two trajectories in a small neighborhood along x are separated by the flow in a time range $[t, t + T]$. This value is calculated as follows: consider the mapping, $\varphi_t^{t+T} : x(t) \to x(t + T)$, which maps the position of the particles from an initial position x at time t to their location at time $(t + T)$. The FTLE with respect to x, t and T is then defined as [8],

$$\sigma_t^{t+T}(x) = \frac{1}{|T|}\log\left\|\frac{\mathrm{d}\varphi_t^{t+T}(x)}{\mathrm{d}x}\right\|. \tag{5}$$

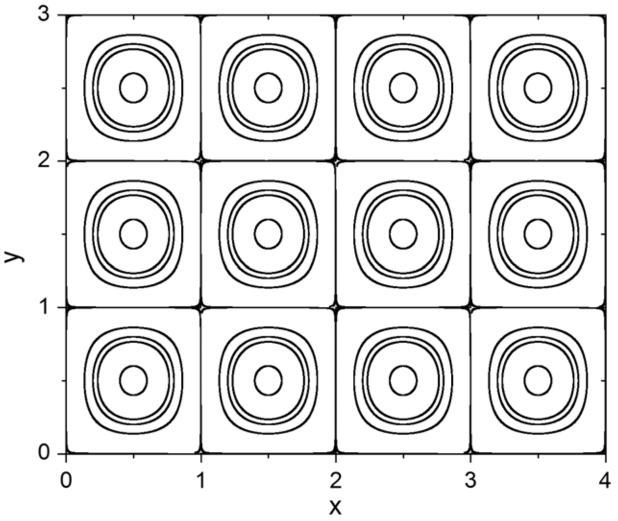

Fig. 1. Phase space of Eq. (3) with $x \in [0:4]$ and $y \in [0:3]$ for $A = 0.1$ and perturbation parameter $\varepsilon = 0$.

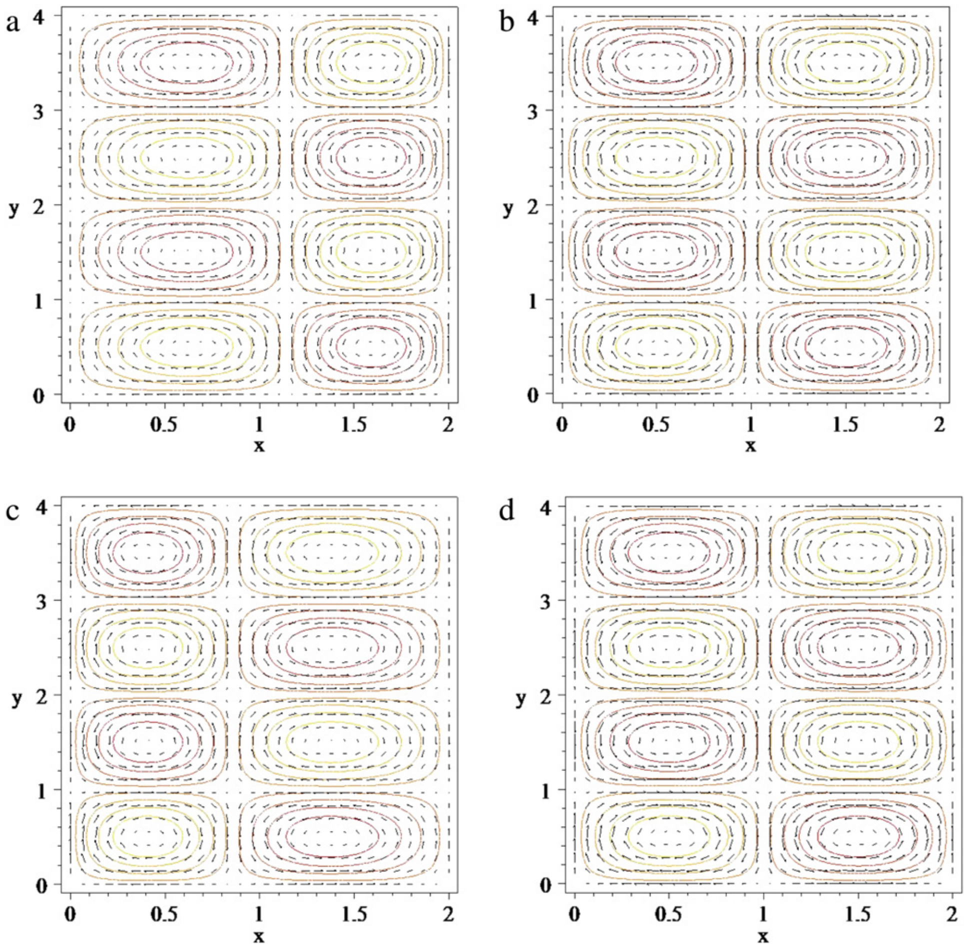

Fig. 2. Velocity fields for different fixed time, with $x \in [0:2]$ and $y \in [0:4]$ and parameters $A = 0.1$; $\varepsilon = 0.25$; $\omega = 0.2\pi$. (a) $t = 1$; (b) $t = 5$; (c) $t = 6$; (d) $t = 10$.

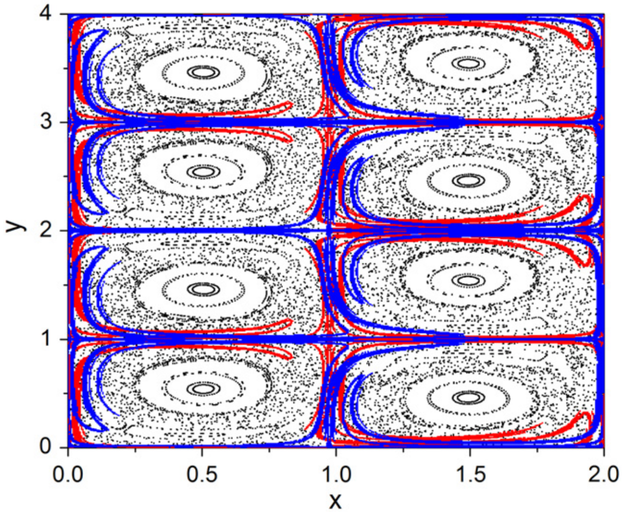

Fig. 3. Poincaré map from Eq. (3) with stable (in blue) and unstable (in red) manifolds, with $x \in [0:2]$, $y \in [0:4]$, $A = 0.1$ and perturbation parameter $\varepsilon = 0.05$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

We used the software MANGEN [16], developed at the California Institute of Technology to calculate the FTLE, for double-gyre flow, with an integration time length $T = 15$. The exponent was calculated in points located in the range of $x \in [0:2]$ and $y \in [0:4]$, for instants from $t = 1$ to $t = 10$. Fig. 3 shows the time-dependent case for a chosen Poincaré section at $(2\pi/\omega)$ and $\varepsilon = 0.05$. We can see that the hyperbolic points are located at the horizontal lines, called robust tori [11–13]. The curves in blue are the stable manifolds and the curves in red are the unstable manifolds from different hyperbolic points. The stable (unstable) manifolds were obtained by calculating the FTLE forward (backward) in time in a grid of points located in the range of $x \in [0:2]$ and $y \in [0:4]$. They intersect each other in some regions forming heteroclinic tangles. When we calculate the FTLE forward in time, two points located in different sides of a stable manifold will separate much faster, due to the exponential divergence they experience while approaching the hyperbolic fixed point. The same occurs with the dynamics around the unstable manifold when we calculate the FTLE backward in time. Then, for $T > 0$, the FTLE for stable manifolds are the highest ones [7].

In Fig. 4 the curves in red are the ones with higher FTLE values. They are stable manifolds defining the flow zones of the velocity fields (Fig. 2) in each cell, representing the first LCS well known in the literature [4]. One can also observe the existence of red lines for every integer y, which are the RT-like LCS. This new kind of Lagrangian coherent structure avoids merging the vortices trapped in different cells. Each cell is made of two RT-like LCS. As we can see in Fig. 4(a–f), the double gyres from different cells do not interact. The two gyres (or two vortices) trapped in a cell only interact along the x-direction, because the RT-like LCS located at integers y are playing the role of robust barriers for the system.

In general, for other systems, we could identify RT-like LCS among other LCS by analyzing the FTLE space. Namely, lines whose points have a maximum finite time Lyapunov exponent act as robust tori-like Lagrangian Coherent Structures.

As already reported for static RT, that appear in autonomous Hamiltonians [11–13], the RT-like LCS can naturally appear in physical systems by choosing the correct parameter combination. In this work we present a particular example, but we believe that this new kind of Lagrangian coherent structure is absolutely generic and can be seen or applied in different branches of science.

4. Conclusions

We demonstrate the existence of robust tori-like Lagrangian coherent structures, from analytical and numerical analysis, composed by barriers that remain intact for any value of the perturbation parameter ε along the time, blocking the interaction of vortices in the space.

This kind of Lagrangian coherent structure can be useful in studies about turbulence in fluids and plasmas, where its detection/creation could be relevant to analyze the transport [8]. This result could be applied as an attempt to detect or create robust tori-like LCS in different systems like the oceanic transport, to identify or create regions that trap materials such as nutrients and phytoplankton [17], and magnetically confined plasmas to improve the turbulent plasma confinement [11–13].

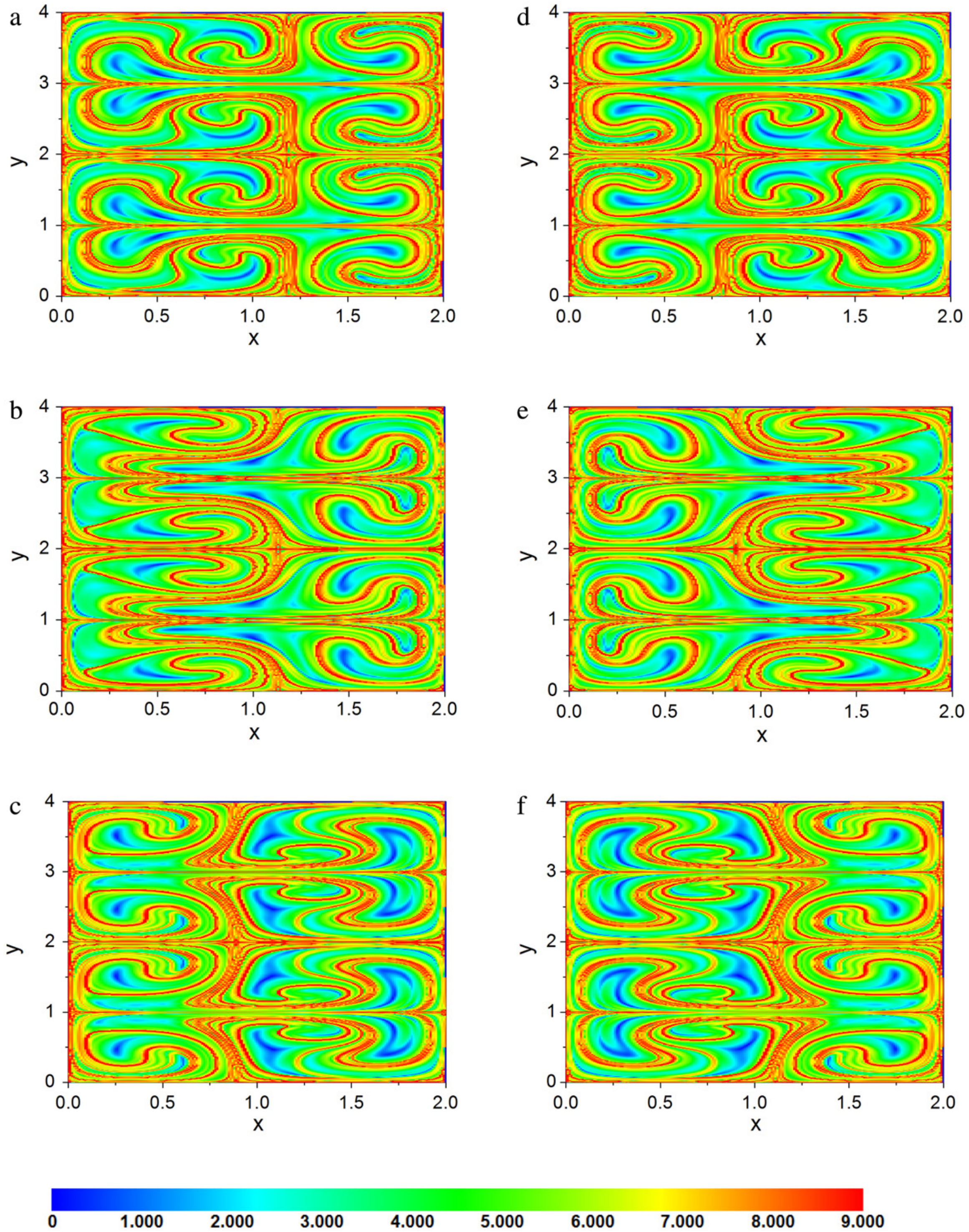

Fig. 4. Finite-Time Lyapunov Exponent (FTLE) for $x \in [0 : 2]$, $y \in [0 : 4]$, the parameters $A = 0.1$; $\varepsilon = 0.25$; $\omega = 0.2\pi$ in the following time: (a) $t = 1$; (b) $t = 3$; (c) $t = 5$; (d) $t = 6$; (e) $t = 8$; (f) $t = 10$. The colors show the values of the FTLE.

Acknowledgments

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